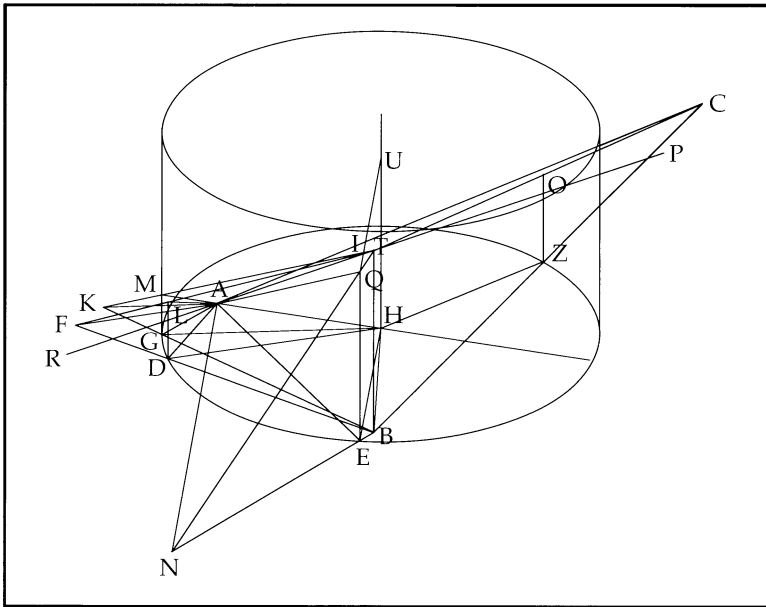


ALHACEN ON THE PRINCIPLES OF REFLECTION

A Critical Edition, with English Translation
and Commentary, of Books 4 and 5 of
Alhacen's *De Aspectibus*, the Medieval
Latin Version of Ibn al-Haytham's
Kitāb al-Manāẓir

Volume Two
English Translation



A. Mark Smith

Transactions
of the
American Philosophical Society
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Volume 96 Parts 2 & 3

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PRINCIPLES OF
REFLECTION**

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VOLUME ONE
Introduction and Latin Text

VOLUME TWO
English Translation

A. Mark Smith

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VOLUME 96, Part 3

Alhacen on the Principles of Reflection:

A Critical Edition, with English Translation
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of Alhacen's *De aspectibus*

VOLUME TWO

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Topical Synopsis

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the point on the axis where the normal through the point of reflection falls. [5.27-39] Every point of the visible portion of a convex conical mirror will be a point of reflection for any given center of sight. [5.28-29] If the center of sight is placed lower than the vertex of a conical convex mirror so that the line of sight forms an acute angle with the axis of the mirror at its vertex, less than half the mirror's surface will be visible. [5.30] If the center of sight is placed so that the line of sight forms a right angle with the axis of a convex conical mirror at the vertex, half the mirror's surface will be visible. [5.31] If the center of sight is placed above the vertex of a convex conical mirror, so that the line of sight forms an obtuse angle with the axis of the mirror at the vertex, more than half the mirror's surface will be visible. [5.32-33] If the center of sight is placed so that the line of sight coincides with a line of longitude on the surface of a convex conical mirror, all of that surface except for the edge along the line of longitude will be visible. [5.34-36] If the center of sight is placed so that the line of sight enters the cone through the vertex, all of the mirror's surface will be visible. [5.37-39] Every point on that surface can therefore be a point of reflection for any center of sight. [5.40-42] The only two ways in which the plane of reflection can cut a convex conical mirror so that reflection can occur is along a line of longitude or along a conic section; if the plane of reflection cuts the cone along a circle, there can be no reflection in that plane. [5.43-45] If the plane of reflection cuts the cone according to a conic section, there can be at most two points of reflection in that plane for any given center of sight. [5.46] In spherical concave mirrors, if the center of sight lies within the sphere from which the mirror is composed, the entire surface of the mirror is visible; if the center of sight lies outside that sphere, more than half the mirror's surface is visible. [5.47] If the center of sight lies at the mirror's center of curvature, then the point on the surface of the eye through which the normal to the mirror's surface passes is all that will be seen. [5.48-49] If the center of sight lies outside the center of curvature, then the form of an object-point can reflect to the center of sight from some point on the mirror's surface, other than the point where the normal from the center of sight falls. [5.50] Accordingly, in every plane of reflection within a spherical concave mirror, there will be a center of sight, an object-point, a point of reflection, and the end-point of the normal dropped from the center of sight through

the center of curvature. [5.51] In concave cylindrical mirrors, if the eye lies within the cylinder from which the mirror is composed, the entire surface will be visible, whereas if it lies outside the cylinder, more than half the surface will be visible. [5.52-53] If the plane of reflection cuts a circle on the mirror's surface, then any point on that surface within the visible portion of the mirror can serve as a point of reflection for any center of sight within that plane of reflection. [5.54] If the plane of reflection cuts a line of longitude on the mirror's surface, then any point on that line of longitude will serve as a point of reflection for any center of sight within that plane of reflection. [5.55] If the plane of reflection cuts a circle on the mirror's surface, any point on that circle will serve as a point of reflection for any center of sight within that plane of reflection. [5.56] If the plane of reflection cuts a conic section on the mirror's surface, then there can only be two points of reflection for any given center of sight within that plane of reflection. [5.57-58] All, none, or more than half the surface of a concave conical mirror can be seen depending on where the center of sight is located; if the plane of reflection cuts a conic section on the visible portion of such a mirror, there can be two points of reflection within that plane. [5.59-60] If the conical mirror is integral, and if the center of sight is situated on the side of its base, nothing outside the mirror can be seen in it, but if the mirror is open, then outside objects can be seen in it. [5.61-62] Closing observations about reflection and image-formation in the various types of mirrors.

[BOOK FOUR]

This book is divided into five parts. The first part [constitutes] the book's prologue; the second [is devoted] to showing that the reflection of light occurs from polished bodies; the third [focuses] on how the [visible] form is reflected; the fourth part [is concerned] with showing that the perception of a form in polished bodies is due solely to reflection; the fifth part [deals] with how forms are perceived through reflection.

[CHAPTER 1]

[1.1] We have already explained in the [previous] three books how [visible] forms are perceived by sight when it is direct [and uninterrupted], and we have listed the particular characteristics of visible objects that the visual faculty perceives.¹ But visual perception occurs in three different ways: i.e., directly, as we [just] mentioned; or by means of [rays] reflecting from polished bodies; or by means of [rays] passing through transparent media, there being none more transparent than [pure] air. These three are the only ways in which vision can occur.²

[1.2] Furthermore, in the case of the latter two [types of vision], the visual faculty perceives all the visible characteristics of objects we discussed earlier when we showed how sight apprehends them in direct vision. And in these cases sight may be subject to error or it may be veridical.³ In this book we shall focus upon how [visible] forms are apprehended by means of reflection, as well as how reflection [itself] will occur, and [we shall also discuss] the disposition of reflected rays. But first we should make some preliminary observations.

[CHAPTER 2]

[2.1] From the first book it is clear that light [emanating] from an illuminated body, whether that body is intrinsically or extrinsically illuminated, shines upon every object that faces it, and likewise color is propagated [to-

ward all facing objects] as long as [that color] is illuminated.⁴ Thus, when a polished body faces an illuminated body, the light from the illuminated body, which is mingled with its color, shines upon that polished body, and the resulting light is reflected along with the color, no matter whether it is bright or faint or whether it is primary or secondary.⁵

[2.2] That reflection may occur in the case of bright light can be demonstrated by directing an iron mirror⁶ toward an intense light-source, having a wall face the mirror, and letting the light shine upon the mirror at a slant, not orthogonally. The bright light will [then] appear reflected upon the wall, but it will definitely not appear at the same spot if the mirror is removed, nor will it appear at the same spot if the mirror is moved; on the contrary, the spot at which the light is reflected upon the wall will shift with the motion of the mirror. So it is clear that reflection occurs in the case of bright light.

[2.3] This can be demonstrated as easily in the case of faint light. Within a room that has a single window that is not too high above the ground, let daylight, but not [direct] sunlight, shine through that window upon some object.⁷ Place an iron mirror near that object, and put a white body near the mirror. On that second white body the light will appear brighter than [it did] when the mirror was absent, and the increase in that light is due solely to reflection from the mirror, for when the mirror is removed, only faint, secondary light will appear on the white body.

[2.4] Furthermore, if one carefully directs his sight along the lines according to which the light shines from the first object to the mirror, he cannot help but realize that the inclination of those lines upon the mirror is the same as the inclination of the lines of reflection to the [white body] itself. So it is characteristic of reflection that the inclination is the same and that the angle of the incident and reflected rays is the same. But if the white body is moved from the location of reflection to somewhere else, even [if that new location is] near the mirror, the increase in light will not appear in it, nor can it be detected except in that particular location [of reflection]. So it is clear that this location is peculiar to reflection.⁸

[2.5] You can observe this same thing in the case of secondary light, if the aforementioned mirror is silver and a third white body lies on the other side of the mirror. Secondary light will in fact appear on the third body, whereas on the second [white] body the light will be brighter than the secondary light [on the third body], and it is obvious that the only reason for this increased brightness is reflection.⁹ Moreover, the reflection of light will be obvious everywhere strong light shines upon a body through any window when a mirror is disposed to face the light and a white body faces the mirror in the way that has just been described.

[2.6] But we shall describe the appropriate location of reflection and the disposition of the rays. It has already been demonstrated in the first book that reflected light follows straight lines, so reflection occurs from polished bodies to a definite spot according to a rectilinear course.¹⁰

[2.7] Now, it is evident from earlier discussions that secondary light emanating from a body that is extrinsically illuminated carries the body's color with it.¹¹ Hence, from every illuminated body or light-source, color mingled with light shines on facing polished bodies and, so mingled, is reflected in a determinate direction.

[2.8] The truth of this point can be ascertained if sunlight shines upon an intensely and beautifully colored body inside a room with only one opening. Stand an iron mirror near that body, and near the mirror place a vessel, in the shape of a goblet, with a white object in it, and set this vessel up at the point of reflection so that the reflected light strikes the white object [inside it]. The color of the body upon which the light shines will definitely appear on the surface of the white body, but that would not happen if the white object were placed outside the appropriate place of reflection.¹² And you will find this proven for such different kinds of colors as sky-blue, red, green, or [any other color] of this kind. So it is clear that the color is sent back [by the mirror] mingled with light, and the appearance of reflected color is [even] more obvious if the mirror is silver.

[2.9] Here, however, is why this demonstration may not be evident—i.e., [the demonstration] that reflected color may be perceived on every body that faces the mirror and presents a white surface to it. As has been asserted earlier, weak colors, even though they shine along with light, are not perceived because the forms that are reflected [in such cases] are weaker than the forms from which the reflection originates.¹³ And this can be shown in the case of light, for when bright light strikes a mirror and is reflected to a wall, the light on the wall will appear weaker than it does in the mirror, and there is a noticeable difference [in intensity] between them.

[2.10] This will become apparent in the same way for weak light. In the case of the original room disposed as it initially was, if the third white body is placed at or near the [original] location of the iron mirror, the light will appear brighter upon that body than it does upon the second one, which would not happen unless reflection weakened the light.¹⁴

[2.11] But someone will argue that the reason for this phenomenon is the iron mirror's black coloring, which, having mingled with the light shining upon the mirror, darkens that light, so, when [the resulting light] is reflected to the second body, it appears faint and dark. On the other hand, in the case of the third [white] body placed at or near the [original] location of the mirror, the light shines [upon it] only from the first body with no

mingling whatever of dark coloring. That this is actually not the case, however, is evident from the fact that, when a silver mirror replaces the iron one, the same thing will be demonstrated.

[2.12] In precisely the same way [as before], the reflected color will be fainter than the color from which the reflection originates. This can be shown for the reflection of color if the white body in the room is placed at or near the location of the mirror. The color will appear stronger on that body than [it does] on the white body placed inside the vessel.¹⁵ The same will be evident if a silver mirror replaces the iron one. Therefore, reflection weakens both light and colors, but colors more than light, according to whichever mirror [is used]. And this happens because colors shine more weakly than light, so they are easily weakened in reflection.

[2.13] Now, when a weak color reaches a mirror, it does mingle with the mirror's color, so when it is reflected it appears faint and dark; and reflected forms are weaker than [the forms] at the point of reflection, and reflection is the cause of weakening.

[2.14] One might claim [in contradistinction] that the weakening of forms in reflection is due solely to their [increased] distance from the source [of radiation]. But it will be made clear that, even when direct light and reflected light are equidistant from their source, the reflected light will still be weaker.

[2.15] Let a shaft of sunlight enter a room through a window, and place an iron mirror that is smaller than the window in the air facing it. Let the rest of the light coming through the window shine upon a white body on the ground, and let the light that is reflected from the mirror shine upon a white body that is raised [above the ground]. When this is set up so that the elevated body and the body lying upon the ground lie the same distance [from the window], the [reflected] light shining upon the elevated body will definitely appear fainter than the light shining [directly] upon the body lying on the ground.¹⁶ And this weakening [of light] can be imputed to no other cause than reflection. The same thing will happen if the mirror is silver.

[2.16] This very point can be demonstrated in the case of color, when sunlight passes through a window into the room [and shines] upon a brightly colored object facing a nearby mirror, and when another hollow object¹⁷ with a white body inside it is positioned to catch the reflection. Within the room place another white body of the same kind as the body in the vessel, and let the distance of this white body from the colored body struck by the light shining through the opening be the same as the distance of the white body in the hollow vessel from that same [colored body] combined with the distance of the mirror from that same [colored body].¹⁸ It can be deter-

mined [on this basis] that the color [appears] weaker on the white body in the vessel than [it does] on the body that lies outside it, even though they are [both] equidistant from their source—i.e., from the colored body. And the reason for this is the reflection that weakens color.

[2.17] Moreover, reflected light is brighter than secondary light, even when the [two kinds of light] lie the same distance from their source. For in fact, when reflected light shines on some object, if another object of the same kind is placed outside the location of reflection, and if it lies the same distance from the mirror [as the first object], the light on the second object will appear fainter than on the first one.

[2.18] The same thing will also be evident in the room [used in the previous experiments] if the mirror is laid upon the ground in a direct line with the opening so that it receives all the light [coming] from the window. The light will be brighter on an object lying at the location of reflection than on another object of the same kind lying outside that location at the same distance from the mirror.

[2.19] In the same way, if [the shaft of] light shining through the window is wider than the mirror, if the [excess] light around the mirror falls upon the ground or upon a white body, and if another body lies as far from the mirror as the object upon which the light is reflected, the light upon that body will appear fainter than the light upon the body [at the location] of reflection.¹⁹

[2.20] The same happens in the case of color. If some object lies as far from the mirror outside the location of reflection as another identical object lies from the mirror at the location [of reflection], the reflected color will definitely appear on the object lying at the location of reflection; [whereas] on the other object no color at all may appear. In fact, if the mirror is iron, almost no color, or none at all, will appear, but if the mirror is silver, some color, albeit very faint, will appear upon the object, but it will be far fainter than [the color] on the object that lies at the location of reflection.

[2.21] It is therefore now evident that the forms of lights and colors are reflected from polished bodies and that they are weakened in reflection. And a form [of light or color] that shines directly [upon an object] will be brighter than one that is reflected [to it] when they share the same source and lie the same distance from it. But a reflected form is brighter than a secondary one when they share the same source or [originate from] sources of equal [intensity] and lie the same distance [from their source].

[CHAPTER 3]

Part three: concerning the way forms are reflected from polished bodies

[3.1] A polished body has an extremely smooth surface, and smoothness consists in the parts of the surface being continuous without many pores. Extreme smoothness exists when there is considerable continuity of the parts of the surface and the pores are few and small; and perfect smoothness consists in the absence of pores and the absence of gaps between the segments [of the surface]. Hence, the polish in a polished [body] consists in the continuity of the parts of the surface with very few and very small pores, and perfect polish consists in absolute continuity of the parts along with the [complete] absence of pores.²⁰

[3.2] Reflection will occur in the case of all polished bodies, even though they may be subdivided into different shapes, and [they are all subject to] the same mode of reflection and share the same specific characteristic[s]: i.e., [that] in every polished surface reflection occurs from every point; [that] whatever point on the surface from which reflection occurs is taken, the line of incidence for any form to that point and the line of reflection [extending from that point] will lie in the same plane as the normal dropped to that point; and [that] these lines will maintain an equivalent situation with respect to [that] normal and will form equal angles [with it]. Now, by "normal" I mean the perpendicular to the plane tangent to the polished body at that point [of reflection], and the two lines [of incidence and reflection] lie along with the perpendicular in the same plane, which falls orthogonally to the plane that is tangent to the polished body at the point from which reflection occurs.²¹

[3.3] If, however, the line along which the form reaches the mirror falls perpendicular to it, the form will reflect [back] along that same line and along no other, and this is characteristic of every reflection from every polished body. Thus, if the polished body is flat, the plane tangent [to it] at the point of reflection will be one and the same as the surface of the body. On the other hand, if the mirror is cylindrical and is polished on either the inside or outside [surface], the contact between the mirror's surface and the plane tangent to it will consist only of a line imagined along the length of the mirror.²² The same holds for a conical mirror, [whether it is] polished on the inner or on the outer [surface]. In a spherical [mirror], be it polished on the concave inner [surface] or on the [convex] outer [surface], the plane of tangency touches at only one point.

[3.4] We will explain, moreover, how this account of reflection can be empirically demonstrated for all mirrors. Take a bronze plaque not less

than 12 digits long that is thick [enough] to be quite rigid, and let it be 6 digits wide.²³ Draw a line right along the lengthwise edge [of the plaque] and parallel to it. Place the point of a compass on the midpoint of this line and draw a semicircle whose radius is the width of the plaque.

[3.5] From the centerpoint [of this semicircle] draw a line perpendicular to the diameter that has just been produced [lengthwise along the plaque]. This line will form a radius that divides the semicircle in half. Along this radius measure off 1 digit [below its upper endpoint] and, with the point of the compass placed on the centerpoint [of the semicircle], draw a semicircle according to the size of the remainder of the [original] radius, according, that is, to a radius of 5 digits.

[3.6] Divide the intermediate portions of the first semicircle into as many parts as you please so that they correspond in kind with the first—i.e., [so that] the first [division corresponds to] the first, the second to the second, and so forth—and draw [straight] lines from the centerpoint [of the semicircle] to the points of division.

[3.7] Then, mark off 1 digit on the [perpendicular] radius on the side of the centerpoint, and draw a line through the point [just] marked off parallel to the diameter of the semicircle, or to the [lower] edge of the plaque, which is the same thing.²⁴ Cut off from the plaque the portion bounded by this line and the radius [of the larger semicircle along the lower edge] to the centerpoint as well as by the first lines dividing the semicircle—i.e., to such lines as lie nearest the [semicircle's lower] radius.

[3.8] Afterwards, cut the plaque around the [circumference of] the larger semicircle so that all that remains is the semicircle. Then cut the plaque below the center, sharpening that spot at the center to a point in such a way that it lies on the same plane as the [larger] semicircle and all the other lines.²⁵

[3.9] Next, take a flat, square block of wood that is broader than the bronze plaque by 2 digits [i.e., 14 by 14], and let its height, or thickness, be 7 digits. Then mark the midpoint [on the top surface] of this block, and from it draw a circle that is a full digit larger than the larger circular segment of the bronze plaque. From the same centerpoint draw a circle equal to the smaller circle on the bronze plaque.

[3.10] Then divide the larger circle into corresponding sections equal to the sections [marked off] on the semicircle of the bronze plaque—i.e., so that the first [section marked off on the bronze plaque's circle] corresponds to the first [section marked off on the circle drawn on the wooden block], the second to the second, and so forth.²⁶ Cut all around the [circumference of the larger circle] on the wooden block so that only the [part bounded by the] larger circle is left; this section will now serve as a template for cutting. Cut out the portion on the block bounded by the smaller circle as well, and

the way to do this is to fit another block to this one so that the line passing through the centerpoint of the former and the centerpoint of the latter is perpendicular to [the top surface] of the latter. Then, fitting a lathe to their centerpoints, form the aforementioned [hollow] circular section. Moreover, the other block should fit firmly so that it remains rigidly in place during the cutting.

[3.11] What will remain of the wooden block, then, is a circular ring that is 2 digits thick, 14 digits across, and 7 digits high, and it should be rounded along its height to form a cylinder. But the lines that divide the circumference of this ring according to the divisions [marked off] on the bronze plaque's semicircular circumference are left [on the portion of the top surface that remains].²⁷

[3.12] From the endpoints of these lines draw lines along the length of the outer surface [of the ring, and draw them] perpendicular to the [top] surface [of the ring], and this can be done as follows. Find a sharply pointed ruler to whose endpoint the lines [on the ring's top surface] are applied, move the ruler around until it touches the outer surface of the ring somewhere on the [ruler's] point. Mark [the points where] the endpoints [of the measuring line touch the surface of the ring], and draw the line [through those points], for that will be the perpendicular you seek.²⁸ The same procedure can be followed for each of the lines of division.

[3.13] Another way of doing this is as follows. Place the point of a compass at the endpoint of [a given] line of division, draw on the outer surface of the ring a semicircle whose radius is the height of the cylinder, and divide it in half. Draw a line from [one] point to [the other] point, and so on for each [line of division]. In the same way, draw the perpendiculars to the endpoints of the lines of division on the inner surface [of the ring].²⁹

[3.14] To continue, on the inner surface [of the ring] mark off points on the [aforementioned] perpendiculars at a height of 2 digits above the [bottom] face [of the ring] that is not subdivided. Through these points draw a circle parallel to the [bottom] face [of the ring] in the following way. Form a flat circular template the same size as the smaller circle on the bronze plaque [i.e., 5 digits in radius], and, up to its centerpoint, cut out from it a triangular section of whatever size you please [whose sides are formed] from two radii and an arc on the circle, which allows you to insert the template manually [into the ring], and fit it up to the points that have been marked off. Place it in this way to those points so that it is parallel to the [bottom] face of the ring, and draw a circle [on the inner surface of the ring] according to [the circumference of] the inserted tablet.

[3.15] Then, at the height of this circle, mark off points at a level of half a grain of barley below it,³⁰ and through the points [just] marked off draw a circle using the [aforementioned] template as a guide. Along this latter

circle scoop out a circular cavity that is 1 digit deep and of the same thickness as the bronze plaque.³¹ And let this cavity lie within the [previously measured-off] altitude of 2 digits so that the latter circle [i.e., the upper one defining the top of the cavity] and the [top edge of the] cavity fall in the same plane.

[3.16] Now, insert the bronze plaque into this cavity, and it will fit into the cavity all the way to the smaller circle [drawn on its surface], since the [difference in] length between the [radius of] the smaller circle and [that of] the larger one is 1 digit, which is also the depth of the cavity.³² Hence, the latter circle and the bronze plaque will lie in a common plane, and the perpendicular lines drawn along the height of the ring [along its inner surface] intersect the lines of division [drawn] on the bronze plaque, and they will fall orthogonally to the [surface of] the bronze plaque. Make sure, however, that the surface of the bronze plaque that is subdivided faces the [upper] face of the ring that is [equivalently] subdivided.

[3.17] Next, on the outer surface of the ring mark a point at a height of 2 digits [from the bottom], and, with the point of a compass placed on the point [just] marked, draw a circle with a diameter of a single grain of barley. With an iron drill whose diameter is likewise a single grain of barley, punch a cylindrical hole [through the ring's wall]. Insert a wooden peg into the hole so that it penetrates to the inner hollow [of the ring] and will [therefore] touch the surface of the bronze plaque. In the same way, drill holes of the same kind and size, at the same height, through each of the perpendicular lines on the outer surface [of the ring].³³

[3.18] Now, take a square wooden block [each] of whose sides is equal to the diameter of the ring [i.e., 14 digits], and through the midpoint of its surface draw a line parallel to its sides and bisecting it. From one end [of that line] measure off a length of 2 digits and mark it off. Then [from that point] mark off a distance equal to the radius of the smaller circle on the bronze plaque [i.e., 5 digits], and placing the point of the compass [at that distance], draw a circle passing through the point [just marked off], that circle being the same size as both the smaller circle on the bronze plaque and the hollow of the [cylindrical] ring.

[3.19] Above the centerpoint of this circle measure off a distance of 2 digits, and do the same below that centerpoint, and mark off the points. From each point draw a line to both sides [of the block] that is parallel to the sides of the square, and on each of these lines mark off a distance of 2 digits on each side from each of the points [just] marked. Then, from the points marked on one of the lines draw parallel lines to the points marked on the other line, and [thus] a square 4 digits on a side will be produced.³⁴ Hollow out this square to a depth of 1 digit, and smooth the sides of the cavity to square them off while likewise making its bottom flat.³⁵

[3.20] Next, attach this block to the [bottom] face of the ring so that the smaller circle [just drawn on it] coincides with the hollow of the ring and so that its edges meet the [outer] circumference [of the ring]. Make this attachment snug with nails so that the block remains perfectly fixed. Bear in mind that the measure of 2 digits applied in all the previous situations must be accurate and definite, so make sure to maintain that measure for each [relevant] edge so that no error arises from a change in measurements.³⁶

[3.21] Next, make a hollow iron tube that is even [in size throughout] and [whose walls are] fairly thick so that it does not readily penetrate [the small holes drilled into the cylindrical ring's wall] and so that it cannot be squeezed out of shape, and let the diameter of its [outer] circumference be 1 grain of barley. Insert the tube into the holes so that when it reaches the inside of the [cylindrical] ring, it will touch the lines drawn on the bronze plaque. And this procedure will be carried out perfectly if the line on the bronze plaque touches the circumference of the tube at the point where the line [that is drawn] perpendicular to the bronze plaque along the height of the ring [on its inner surface] passes through the center of the circular section of the tube.

[3.22] Attach a ring or a bolt to the end of the tube so that the tube is allowed to penetrate [into the hollow of the ring] only to a determinate point. But make the tube long enough that, as it passes over the bronze plaque, it reaches the line that is parallel to the diameter of the plaque, these two lines defining the [lower] section [of the plaque]. This line is [thus] parallel to the base of the triangle [inscribed] on the bronze plaque.³⁷

[3.23] Then, make seven iron mirrors: one of them plane; two spherical, of which one is concave and polished on its inner surface, the other [convex and polished] on its outer surface; two cylindrical, of which one is concave, the other polished on its [outer, convex] surface; and two conical, of which one is polished on its [outer, convex] surface, the other on its [inner] concave surface. Let the plane mirror be circular, and let it be 3 digits in diameter.

[3.24] Let the cylindrical mirror that is polished on its [outer] surface be clear and highly polished, and let the diameter of the circle at its base be 6 digits. Furthermore, let the cylinder be 3 digits high. At the base of the cylinder measure off a chord 3 digits in length. Likewise, at the opposite base of this same cylinder measure off a chord of equal length directly facing the other chord so that the lines drawn from the endpoints of one chord to the [corresponding] endpoints of the other are perpendicular [to both bases]. Cut the cylinder along these lines so that what we have left is a portion of the cylinder whose bases consist of the sections of those chords, or [to put it another way, so that] the height of the remaining portion's axis

is less than half a digit. By "axis" I mean the line [extending] from the midpoint of the arc [at the base] to the midpoint produced on the chord.

[3.25] Let the height of the concave cylinder be 3 digits, let the diameter of its base be 6 digits, measure off a chord of 3 digits on that base, and cut off a section as [you did] in the first [cylinder]. So the height of the remaining portion's axis will be less than half a digit. Make sure that all these mirrors are exquisitely polished and even throughout.

[3.26] Find a conical mirror whose base circle is 6 digits in diameter, the chord at its base being 3 digits, and let it be 4.5 digits high along the longitude. Cut out the [appropriate] portion along the straight lines [connecting the endpoints of the chord to the vertex of the cone] so that the axis of the [base] of the remaining portion is less than half a digit long, and see that this [is done] in each cone [i.e., concave and convex].

[3.27] Let the [convex] spherical mirror be a portion of a sphere with a diameter of 6 digits, and let the base [of the cut-off portion] of the mirror be 3 digits in diameter, so the axis [of this portion] will be less than half a digit. The same should be done in the case of the concave spherical mirror.³⁸

[3.28] Then you should produce seven flat wooden panels whose sides are parallel and orthogonal [at the corners] so that the edges are as parallel as possible, and the panels should be 6 digits long and 4 digits wide. Next, fit [one or] another of the panels into the hollow square [in the block attached at the base of the cylindrical ring] so that it stands perfectly erect on the bottom of the hollow square, and see to it that it fits easily into the [hollow] square without being squeezed out of shape.

[3.29] Accordingly, let the point on the bronze plaque touch the face [of the panel], mark off the point where it will reach to it, and from the point [just] marked to the [upper and lower] edges of the panel draw a line parallel to the sides of the panel so that this line forms a line of longitude on the panel.³⁹ Then, on the longer segment of that line, from the point just marked off, measure a distance of half a grain of barley, and mark the point. I say that this point lies at the midpoint of the panel and that it also lies directly in line with [each] centerpoint of the holes [in the wall of the ring].

[3.30] [Here is] the proof. Since the centerpoints of the holes lie half a grain of barley above the surface of the bronze plaque, and since those centerpoints lie 2 digits above the [bottom] surface of the ring, then that point lies 2 digits above that same surface. But the panel is sunk to a depth of 1 digit in the hollow square [in the block attached to the bottom of the ring]. Since the distance between the [top and bottom] edges of the panel and the point is 3 digits, then that point constitutes the midpoint. At this midpoint draw a line from side to side across the panel and parallel to the [top and bottom] edges. Then bisect the [two remaining sections of the] line of longitude to which this line is perpendicular with orthogonal lines

that are parallel to the [top and bottom] edges. Hence the panel will be subdivided into four equal portions. Do the same thing with the [six] other panels.⁴⁰

[3.31] When all this is finished, fit the plane mirror into one of the panels. In order to do this, scoop out a hollow [in the panel] as deep as the thickness of the mirror so that the surface of the mirror lies in the same plane as the surface of the panel, the midpoint of the mirror's surface lies directly upon the midpoint of the panel's surface, the line bisecting the surface of the panel also bisects the surface of the mirror, and the [top and bottom] points of the mirror coincide with that dividing-line. Be as careful as possible to follow this procedure accurately.⁴¹

[3.32] Then fit the cylindrical mirror that is polished on its [outer convex] surface into one of the panels so that its midpoint coincides with the midpoint of the panel, the line taken along the length of the mirror that bisects it coincides with the [remaining] segments of the line along the length of the panel that bisects its surface, and the midline along the length of the mirror lies in [the plane] of the panel's surface. And this can be accomplished if the arcs at both bases of the mirror are bisected, and a line is drawn from one point of bisection to the other. Match this line to the midline along the length of the panel so as to coincide with it.⁴²

[3.33] Fit the concave cylindrical mirror into a panel so that the midline taken along its length, which bisects the arcs at the bases, is parallel to the midline along the length of the panel and so that the chord on each arc, along with the parallel lines along the mirror's edge, lies in [the plane of] the panel's surface.⁴³

[3.34] Fit the conical mirror that is polished on its outer [convex] surface into a panel so that its vertex lies at the end of the midline along the length of the panel, and the line bisecting the section of the cone [from which the mirror is formed]—i.e., the line that extends from the vertex to the midpoint of the arc at its base—lies in [the plane of] the panel's surface and coincides with the remaining segment of the midline along the length of the panel.⁴⁴

[3.35] Fit the concave conical mirror into a panel so that its vertex lies right at the midline along the length of the panel, and let the chord of the arc at its base lie in [the plane of] the panel's surface. The line drawn from the vertex to the midpoint of the arc at the base [of the mirror] should be parallel to the midline along the length of the panel. Furthermore, since the cone is 4.5 digits long, 1.5 digits will be left along the longitudinal midline of the panel [on its bottom half].⁴⁵

[3.36] In order to fit the spherical mirror that is polished on its outside [convex] surface into a panel, draw a circle 3 digits in diameter on the panel. Let its centerpoint be the centerpoint of the panel. Then scoop out [a hol-

low according to that circle], and fit the mirror into it so that the midpoint on its surface lies in the plane of the panel[’s surface] and coincides with the midpoint of the midline along the panel’s length. One can make sure that this is done correctly by applying another sharp-edged ruler of the same length [as the panel] and identically subdivided upon the midline along the length of the panel so that the midpoint of this sharp-edged ruler touches the midpoint of the spherical mirror.⁴⁶

[3.37] Having drawn a circle 3 digits in diameter on a panel, the centerpoint of that circle being the panel’s centerpoint, and having scooped out this circular section, fit the concave spherical mirror into [the resulting cavity] so that the circle at the base of the mirror lies in the plane of the panel[’s surface], and so that the midpoint of the concave surface of the mirror is directly opposite the panel’s midpoint. To ensure that the diameter of the mirror’s base coincides with the midline [along the length] of the plank, do the following. Mark a point on [the edge] of the sharp-edged ruler, and on each side of that point measure off a distance equal to the radius of the [circle forming the] mirror’s base. Accordingly, apply this sharp-edged ruler to the midline of the panel so that the point marked on it lies directly opposite the midpoint of the concave surface of the mirror and so that the diameter marked off on it is the same as the diameter of the [mirror’s] base.⁴⁷

[3.38] Once all this is accomplished, measure off from the endpoint of the radius that bisects the triangle on the bronze plaque at its sharpened point a distance equal to the axis of this concave mirror [i.e., ca. .4 digits], and mark this point. The axis, moreover, can be determined as follows. Place a sharp-pointed ruler on the surface [of the panel bearing the mirror] so that its sharp point lies directly on the midline along the panel’s length and directly above the midpoint on the concave surface [of the mirror]. Then, from that point on the ruler, lower a thin, sharp needle perpendicular to the mirror. It will, of course, reach to the centerpoint of the concave [surface of that mirror]. On the needle mark the point where, after it has reached [the mirror’s surface], it touches the point of the ruler or a point [previously] marked [on the ruler], and slant the ruler slightly so that the mark can be made accurately on the needle. Then, from the vertex of the triangle on the bronze plaque and along the line that bisects that triangle, measure off the distance from the needle’s point to the point [just] marked on it, and mark that point [on the triangle’s line of bisection].⁴⁸

[3.39] Next, you should insert the panel [containing the concave spherical mirror] into the hollow square [at the bottom of the cylindrical ring] so that the point of the bronze plaque touches the mirror; [against the face of the panel] apply the sharply pointed ruler [orthogonally] to the line that

bisects the triangle [on the bronze plaque] so that a point may be marked on that line, which is touched by that sharply pointed ruler, since the vertex of the triangle reaches all the way to the surface of the concave mirror. Accordingly, mark that point.

[3.40] This second point, however, will lie a smaller distance from the vertex [of the triangle] than the first point, because the surface of the bronze plaque lies 2 digits minus half a grain of barley from the [bottom] surface of the [cylindrical] ring, or from the [top] surface of the block containing the square cavity. On the other hand, the midpoint of the panel [containing the mirror] is directly opposite the midpoint of the concave spherical mirror, that point of course lying 2 digits above the same surface [i.e., the top surface] of the block [at the base of the ring]. Consequently, since the vertex of the bronze plaque extends orthogonally [to the panel containing the mirror], it will not reach the midpoint of the concave [surface], which is the end of the [mirror's] axis, but to a point that is higher [with respect to the axial height of the mirror], so [we have established] what we set out [to show].⁴⁹

[3.41] Mark the spot on the concave mirror where the vertex of the bronze plaque reaches, and, having bored a hole at that point, move the vertex orthogonally [into the hole] just far enough that the sharp edge of the ruler that is applied [to the face of the panel] touches the point first marked on the line bisecting the triangle. That being so, the point of the bronze plaque will lie in the same plane as the endpoint of the mirror's axis [on the convex surface of the mirror], and that plane is parallel to the surface of the panel [containing the mirror]. And the line drawn from the endpoint of the axis to the point [of the triangle] will be perpendicular to the surface of the bronze plaque.⁵⁰ Moreover, the axis of the mirror lies in the same plane as the centers of the holes, because they lie 2 digits from the [bottom] surface of the [cylindrical] ring, and so does the endpoint of the axis [at the] midpoint [of the mirror].

[3.42] When all of this is carefully done, what we predicted can be empirically verified. Insert the panel with the plane mirror upon it into the ring until the point of the bronze tablet touches the mirror, fix the panel into the hollow square [at the bottom], and apply something [adhesive] to the bottom of the panel in order to keep it firmly nested so it does not wobble. Then press [a piece] of parchment up to the holes [in the wall of the cylindrical ring], and make an impression [of each hole] with the finger in order to fill in the holes so that you can make out the impression. Then mark the impression of [each] hole on the parchment with red ink or something else [of that kind]. Leave one of the holes open, however, but not the one directly facing the middle of the tablet, and point the open hole at an [incoming] beam of sunlight. The result of this operation will be clearer if the

apparatus is held up to a ray of sunlight entering a room through a window.

[3.43] Accordingly, when the beam passing into the hole reaches the mirror, you will see it reflected to the corresponding hole [on the wall of the cylindrical ring] along the line on the bronze plaque that forms with the line bisecting the triangle an angle equal to the angle formed by the line from the open hole with the same radius [that bisects the triangle]. On the other hand, if you uncover the hole to which the ray was previously reflected and shine the light through it, you will see the ray reflected to the [previously] open hole.⁵¹

[3.44] Furthermore, if into the hole you insert the hollow iron tube that we made earlier according to the size of the hole (apply a bit of wax around it to make it nest snugly), the light will pass through the hollow of the tube just as it passes through the hole. And it will be reflected to the corresponding hole, and the incidence and reflection will follow the [corresponding] lines on the bronze plaque as before. Also, if we transfer the tube to the second hole, we will see the light reflected to the first one. However, the light passing through the tube will be weaker than it was when it passed through the hole without the tube in it. Still, the same way of reflecting will be observed in the case of the weaker light [as in the case of the more intense light].

[3.45] Block the tube with wax in such a way that a tiny hole remains at its center, and the light will be seen to reflect to [the corresponding] hole, or, rather, to its centerpoint. Likewise, if you fill the hollow of the tube with wax in such a way as to leave [a tiny channel] virtually the size of the [tube's] axis, the light will pass along the axis of the tube and will be reflected to the centerpoint of the corresponding hole. By the same token, if the tube is inserted into the other hole, then when the light passes along the axis of the one hole, it will be reflected along the axis of the corresponding hole, for the center of the hole lies directly in line with the axis.⁵² And since the reflected light falls at the center and only radiates along a straight line, it follows that it must proceed along the axial line.

[3.46] Next, after blocking each of the holes except the middle one [whose axis] continues directly along the [central radius of the] bronze plaque, make a cylindrical pointer the size of the hole, and sharpen its end so that it comes to a point at the end of its axis. Insert it through the hole and mark the spot on the mirror where it touches. Then let a beam of sunlight shine through that hole. It will in fact fall upon the marked point, and it will form a circle around it.

[3.47] Now, mark a point on the circumference of this circle of light, and draw a circle whose radius is the size of the line joining the points just marked off. This circle will in fact be larger than the circle of the hole, for light

shining through a hole propagates in the form of a cone. But the light will be seen reflected to none of the [other] holes, so it is obvious that the light shining through the axial hole is reflected back to the same hole. Nevertheless, the circle of light formed at the inner base of the hole will appear larger than the ray itself [upon entering into the ring], and it will also appear larger than the circle of light [cast on the mirror] inside [the ring].⁵³

[3.48] So it is obvious that the way this light appears is due to reflection, but not to the reflection of light passing along the axial line, a fact that can be demonstrated as follows. If both bases of the hole are blocked so that all that is left is a narrow opening along the axis, and if a ray of sunlight shines along the axial line, that light will not appear to form a circle around the interior base of the hole because it was not formed by light reflected along the axis.⁵⁴

[3.49] Earlier, moreover, we set up [this] particular panel so that it stood orthogonally on the [bottom of the] hollow square. If the conditions [under which we set it up] are slightly changed so that the panel may be slanted in such a way that the end farther from the square is inclined down toward the ray passing through the middle hole, the ray will not fall orthogonally upon the mirror, so the light will appear reflected away from the middle hole. And the greater the slant, the farther away from the hole the reflected light will be. But if the panel is restored to an upright position, the reflected light will appear around the inner base of the hole as it did before.

[3.50] It is therefore evident that, when light falls orthogonally upon the mirror, it returns back to the hole through which it entered. When, however, the light falls along a line slanted to the axis, it does not reflect to the hole but will appear upon the line on the [inner] surface of the ring that is perpendicular to the bronze plaque and that passes through the centerpoint of the middle hole.⁵⁵

[3.51] Furthermore, everything that has been said in the case of the first pair of holes that are [correspondingly] inclined you must understand [to hold] in the case of every [other pair of such holes]. Bear in mind, as well, that what has been claimed for the plane mirror [applies to] the other [types of] mirrors, whether the light passes through an inclined hole or through the middle hole, or whether the panel is [held] upright or at a slant.

[3.52] If, however, the panel containing the cylindrical mirror that is polished on its outer [convex] surface is held at a slant in the [hollow] square so that it does not stand upright in the square but is inclined to the right or left, the light will still appear to be reflected upon the hole that corresponds to the hole through which it enters, and the light [passing through] the middle [hole will be seen reflected back] to the middle hole, just as was seen when the panel was not slanted.⁵⁶

[3.53] You should [next] insert the panel with the concave cylindrical mirror in it, and let the point on the bronze plaque approach it until it touches the surface of the mirror. You will slant this mirror to the side just as you slanted the one polished on its outer [convex] surface.

[3.54] You will follow the same procedure for concave conical mirrors.

[3.55] Set up the concave spherical mirror so that the point of the bronze plaque enters the hole in the mirror that was made according to the insertion of that point.⁵⁷

[3.56] The spherical mirror that is polished on its outer [convex] surface should be inserted so that the point of the bronze plaque reaches the plane of the panel and lies in the same plane as the midpoint of the mirror. This can be done as follows. Apply the sharp-pointed ruler to the panel so as to touch the midpoint of the mirror, and move its point down to the bronze plaque until that point is directly in line with the point of the ruler. Then bring them together.⁵⁸

[3.57] In the case of cylindrical mirrors, you will observe [what happens in] reflection as follows. Set up the mirror as described before, and pass the cylindrical pointer through the middle hole, just as was done in the case of plane mirrors. Of course the pointer will fall upon the midline of the mirror along its length, and its point will lie in the plane of the panel. On this midline mark the point where it falls, and from this point measure off on the surface of [the mirror inserted in] the panel the distance of the radius of the circle drawn [earlier] on the panel for the purpose of observing how the impinging light forms a circle [on the mirror's surface]. Measure off the same distance on the other side of the point, and a line equal to the diameter of the aforementioned circle will be determined. Moreover, the impinging light will be observed to extend only upon the aforementioned line, and it is reflected to the middle hole. And the circle of light at its inner base will appear larger than the circle of light [on the mirror] inside [the ring], just as was observed in the case of plane mirrors.

[3.58] You can observe the same thing in the case of conical mirrors.

[3.59] Likewise, in the case of spherical mirrors, when light passes [to them] through the middle hole, draw a circle on the surface of the [the mirror inserted in the] panel the same size as the aforementioned circle. The light will be observed to fill this circle and then reflect to the middle hole in the way already described. In all these straight-on reflections the perpendicular line [of longitude] drawn along the inner surface of the [cylindrical] ring will appear to transect the circle of reflected light and divide it in half.

[3.60] What has been described for natural [or primary] light can be observed for accidental [or secondary] light. In a room with one window place a screen facing [the window] so that a beam of sunlight shines upon

it, and set up the apparatus with respect to the window so that the accidental light [radiating from the screen] passes through one of the holes, but not the middle one. The light will be seen to reflect to the [corresponding] hole opposite it, and if the instrument is set up so that the light enters through two holes, it will be reflected to the two corresponding holes.

[3.61] Moreover, in order to be able to determine that the light enters directly and passes straight through, place the aforementioned piece of parchment [inside the cylindrical ring], and turn the apparatus until you see the light shining on the parchment. In fact, the shining of accidental light on mirrors is not clearly perceived because of its faintness. However, in this kind of light the same thing will be evident as was evident in natural [i.e., primary] light, for there is no difference in their nature except that the one is bright, the other faint.

[3.62] Thus, it is clear that lights reaching mirrors along various lines are reflected along various [corresponding] lines. And if light reaches the mirror from the same direction, it continues [from it] in an equivalent direction, and the inclination of the reflected rays will be equal to the inclination of the incident rays. And it is evident that the incident and reflected rays of light lie in the same plane, which is perpendicular to the polished surface or [the surface] tangent to the point from which reflection occurs. Moreover, if the light arrives along the perpendicular, it will reflect along the perpendicular, and no matter what point it strikes, it is reflected in a plane that is perpendicular to the plane tangent to that point.

[3.63] In addition, the reflected ray invariably forms with the normal to that point an angle equal to the angle formed by the incident ray with that same normal. And the proof of this point is evident from the fact that, if any light shines through any hole [in the ring], it is reflected to the corresponding hole. And if the hole is narrowed so that all that is left is virtually equivalent to the axial line, the light is reflected along the axis of the corresponding hole. And if the light shines through the other hole, it is reflected along the lines according to which it was incident before. And it is obvious that the corresponding holes are equivalently disposed with respect to the middle hole, and since light only propagates along straight lines, it is evident that it is reflected along lines that are equivalently disposed with respect to the middle hole as the lines of incidence.

[3.64] Hence, when it arrives along the orthogonal, it is reflected along that line alone, because the lines of reflection invariably maintain the same disposition as the lines of incidence with respect to the plane that is tangent to the point of reflection. And this is [an] essential characteristic [of light], whether the light be essential or accidental, or whether it be intense or faint, and it applies universally in every case.⁵⁹

[3.65] Now we shall demonstrate the equivalent disposition [of the lines of incidence and reflection]. We already know that the surface of the panel stands orthogonal to the [bottom of the] block [at the base of the ring] in which we dug out the [hollow] square. Hence, the midline of the base-block is perpendicular to the common section formed by the [top surface of the] base-block and the panel, and it is [therefore perpendicular] to the line across the width of the panel [formed by this common section]. Moreover, the [top surface of] the base-block is parallel to the [top surface of the] bronze plaque, and its midline is parallel to the midline of the bronze plaque, that is, to the line drawn from the center of the bronze plaque [at the point of its triangle] that bisects its arc.

[3.66] Furthermore, the common section of the [top surface of the] bronze plaque and the panel, which is a line across the width of the panel, is parallel to the common section of the base-block and the panel, so the midline of the bronze plaque falls orthogonally to the common section of the panel and the bronze plaque. And the panel stands perpendicular to the [bottom] surface of the square [hollowed out of the base-block], and the [bottom] surface of the square [hollowed out of the base-block] is parallel to the [top] surface of the base-block [itself], so the [top] surface of the base-block is orthogonal to the surface of the panel.

[3.67] Likewise, the surface of the bronze plaque stands perpendicular to that same surface [i.e., of the upright panel], and the midline along the panel's length is perpendicular to the line across its width, so the midline of the base-block [at the bottom of the ring] will be perpendicular to the midline along the length of the panel, since it stands [upright] upon it; and, by the same token, the midline of the bronze plaque will be perpendicular to that same midline. Accordingly, the midline of the bronze plaque is perpendicular to the surface of the panel as well as to the midline along its length, and so it is perpendicular to the surface of the plane mirror as well as to the midline along its length.⁶⁰

[3.68] In addition, the surface of the bronze plaque is parallel to the plane passing through the centers of the holes [in the ring], for the centers of [all] the holes lie the same distance from the surface of the bronze plaque—i.e., half a grain of barley—and the diameter of [each] hole is 1 grain of barley. Likewise, the diameter of the surface of the [iron] tube is 1 grain [of barley], and the plane passing through the centers of the holes bisects the tube. Hence, the axis of the tube lies in that plane, and, as it extends [into the hollow of the ring], the tube touches a line [drawn] on the [surface of the] bronze plaque, that line of course being parallel to the [tube's] axis, for that axis is parallel to any line [of longitude] on the surface of the tube.

[3.69] Moreover, the axis of the tube falls to a point on the surface of the panel, and the line drawn from that point to the center of the bronze plaque

is perpendicular to the bronze plaque, because, no matter what hole the tube extends through, its axis falls upon the midline along the length of the panel, and all of these perpendiculars are equal [in length].

[3.70] Also, the line extended from the point where the axis falls on the panel through the center[s] of the holes is parallel to the line extended from the center of the bronze plaque to the endpoint of the hole's diameter. For the line [drawn] between that point [on the panel's midline] and the [bronze plaque's] center is perpendicular to the surface of the bronze plaque, since it is a segment of the midline along the length of the panel, and [it is] also [perpendicular] to the axis [of the hole]. And [every] line on the [inner surface] of the ring that passes through the centers of the holes and falls perpendicular to the surface of the bronze plaque is parallel to this line, which extends between the centerpoint of the bronze plaque and that point [where the axes of the holes intersect the midline along the length of the panel]; hence, the lines extending from the [respective] endpoints of the [segment of the] line on the [inner surface of the] ring and [the endpoints of the segment of the midline] along the length of the panel will be parallel [since they connect the endpoints] of equal and parallel [lines].

[3.71] The same holds for every one of the holes, because the lines drawn from the point on the panel where the axis falls to the centers of [any] pair of corresponding holes are parallel to the two lines drawn from the center of the bronze plaque to the endpoints of the diameters of the same holes, so the angle formed by these two lines is equal to that formed by the other two.

[3.72] If a line is erected from the endpoint of the axis [where it intersects the midline along the length of the panel] to the center of the [middle] hole, it will lie in the plane passing through the centers [of all the holes], and it will be parallel to the midline of the bronze plaque. For the line connecting the inner ends of these lines is perpendicular to the bronze plaque, and it is equal to the line connecting their outer ends, which is perpendicular to the bronze plaque. And [this inner line] is parallel to that [outer one], so the line from the center of the middle hole to the endpoint of the tube's axis [where it intersects the midline along the length of the panel] is parallel to the midline of the bronze plaque, and that line is perpendicular to the panel, so the other is too.⁶¹ Therefore, this line [extending from the center of the middle hole to the midline along the length of the panel] and the sides forming alternate angles [with it] are parallel respectively to the midline of the bronze plaque and to each of the [corresponding] lines on the [surface of the] bronze plaque forming the [same] angle, so the corresponding segments [of the entire angle formed by those respective sides] are equal.

[3.73] Hence, the midline of the bronze plaque bisects the [entire] angle in its plane, so the [axial] line [drawn] from the center of the middle hole

bisects the angle in its plane.⁶² And since there is no doubt that the light entering one of the inclined holes proceeds along lines that form an angle, it is evident that all light is reflected along lines that lie on the same plane as the lines of incidence, that plane being orthogonal to the reflecting surface [formed by the panel's face], and such lines [of reflection] form an angle with the normal that is equal to the angle formed by the lines of incidence with that normal.

[3.74] Furthermore, light that falls along the normal is reflected along the normal. And this holds universally for all light.

[3.75] However, if the panel is slanted not sideways but [lengthwise] from the top so that the axis of the middle hole is not perpendicular to the panel, the light is [still] reflected, and it will be seen on the line drawn lengthwise [on the inner surface] of the ring, this line being perpendicular [to the bronze plaque] and passing through the center of the hole. And the greater the slant, the farther the reflected light will fall from the hole or from its axis. However, if the slant is decreased, the distance [between the center of the hole and where the reflected light falls] will decrease, so when the panel is restored to a perfectly upright position, the light is reflected along the normal.

[3.76] Furthermore, it is evident from the following that in the case of such an orientation the axis of the middle hole and the line of reflection lie in the same plane, which is orthogonal to the reflecting surface [formed by the panel's face]. For the axis of the middle hole is perpendicular to the [line along the] width of the panel—i.e., to the common section of the panel's surface and the plane passing through the centers of the holes—and the midline of the block [at the bottom] of the ring is parallel to this axis and parallel to the midline of the bronze plaque.

[3.77] Also, the midline of the bronze plaque is perpendicular to the [plane] of the panel along its width, and [so] it is perpendicular to the common section of the panel's surface and the surface of the bronze plaque, so the plane in which the midline of the bronze plaque and the axis of the middle hole lie is orthogonal to the panel's surface. Moreover, the line drawn lengthwise [and perpendicular to the bronze plaque on the inner surface] of the ring lies in this same plane, for it passes through the endpoints of parallel lines—i.e., the midline of the bronze plaque and the axis of the middle hole.

[3.78] It follows, therefore, that the reflected light appearing on the perpendicular drawn lengthwise [on the inner surface] of the ring is reflected along a line that lies in the same plane as the axial line along which it is incident, and that plane is orthogonal to the surface of the panel. Hence, when light is incident upon a plane mirror, its reflection occurs along lines

that are identically inclined to the surface of the mirror, and those lines lie in the same plane as the normal, that plane being orthogonal to the surface of the mirror.⁶³

[3.79] In the case of the cylindrical mirror polished on its outer [convex] surface, the proof is precisely the same as in the case of the plane mirror—i.e., because the point of the bronze plaque falls orthogonally to the [mid]line along the length of the mirror, and so does the [axial line through the iron] tube when it reaches it. And the segment of the [mid]line [of the mirror] that passes through those [points] is orthogonal to the bronze plaque. Invariably, then, whether the light shines through the middle hole or through one of the inclined holes, its [line of] reflection will lie in the same plane as its [line of] incidence, that plane being orthogonal to the plane that is tangent to the [mid]line along the length of the mirror.

[3.80] In the case of the conical mirror [polished] on its outer [convex surface], since the surface of the panel lies in the same plane as the midline along the length of the conical mirror, as it does in the case of the [convex] cylindrical mirror, the lines in the planes [of the axial lines and the bronze plaque] will be equivalently disposed, so reflection will occur in the same way as it does in the plane mirror, and the demonstration will be precisely the same.

[3.81] In the case of the concave cylindrical mirror, as well, the point of the bronze plaque falls on the midline along its length, and the axis of each hole falls on that same line. And the line-segment between these [two points] to which [the two lines] fall is orthogonal to the surface of the bronze plaque, and the axis of the [middle] hole and the midline of the bronze plaque are orthogonal to the plane tangent to that mirror upon the [mid]line along its length, which is where reflection occurs, and they are parallel to the surface of the [bronze] plaque.

[3.82] Thus, as before, [we follow] the same method for proving that [the lines of] reflection and [the lines of] incidence lie in the same plane, which is perpendicular to the reflecting surface [formed by the panel's face], that those lines are equivalently slanted [with respect to the normal dropped to the point of reflection], and that [light] incident through the central hole is reflected back to that hole. Also, when the top edge of the panel is slanted [backward], the light will be reflected to the perpendicular [drawn lengthwise on the inner surface] of the ring, just as was pointed out in the case of the plane mirror.

[3.83] In the case of the concave conical mirror, the same proof holds in all respects.

[3.84] In the case of the [convex] spherical mirror [polished] on its outer surface, it is clear that its midpoint lies in the plane of the panel, while its

axis falls on that point, so the lines [of incidence and reflection] will have the same disposition in that case. And what holds for the plane mirror holds in all respects for the other mirrors, and the demonstration is the same [for all].

[3.85] In the case of the spherical concave mirror it has already been determined that the axis of the hole passes to the mirror's midpoint, and the point of the bronze plaque passes into the hole in the mirror, which we made earlier, until it lies in the same plane as that midpoint. And the line drawn from that point to the point [of the bronze plaque] is parallel to the midline along the length of the panel, and so [the lines of] incidence and reflection lie in a plane that is orthogonal to the plane tangent to the mirror at that midpoint, [that latter plane being] parallel to the surface of the [bronze] plaque. And in this case the demonstration is precisely the same as it is for the other mirrors.

[3.86] Hence, it is evident that if any light shines on any of those mirrors [just discussed], the [lines of] reflection and incidence lie in the same plane, which is orthogonal [to the surface of the mirror or the plane tangent to the point of reflection on the mirror]. But the reason reflection follows this rule is not specific to the axis, or to the point on which the light shines, or to the hole through which it shines, or to the mirror. Indeed, it holds for any hole, no matter what kind of light [shines through it], and it holds for any line of incidence as well as for any point on the mirror to which the light may fall. For, no matter what point on the mirror is taken, if light shines on it, since its disposition is the same with regard to the [midline along the] length of the mirror, and since any of the others will be equivalently disposed with regard to the lines extended from it, those lines are identically inclined with respect to the lines understood [to extend] from that point, just as from the point taken earlier, as well as from any other point.

[3.87] So it is invariably the case that the disposition of each [such line] at the point to which the light falls, which is the same as the previously taken point, is equivalent with respect both to the axis and to the point of the bronze plaque. And the same proof and the same demonstration hold for all cases, so it is certain that this is due not to a particular kind of light nor to the shape of any particular mirror, but it is a characteristic that is common to every polished body and to any kind of light. Moreover, when the light shines through various holes to a given point, the variation in reflection and in the angles of reflection will be observed to conform to the way the light is incident, and the same holds in all cases.

[3.88] It is obvious from what we have established above that, if a polished body faces a luminous body, light from any point on the luminous body falls to any point on the [exposed surface of the] polished body, so at

any point on the polished body there stands a cone whose vertex lies on that point and whose base is formed by the surface of the luminous body. Also, from any point on the luminous body there extends a cone whose vertex lies at that point and whose base is formed by the [exposed] surface of the polished body.

[3.89] Moreover, if some point is imagined between the luminous and polished bodies, the light from the luminous body will reach that point in the form of a cone whose vertex lies at that point, and when the edges of that cone are extended until they reach the surface of the polished body, they form a cone. Hence, the vertices of two cones will lie at the point [so] imagined, their bases consisting of the surface of the luminous body and the surface of the polished body, and if a cone is imagined at some intermediate point with its base formed by the polished surface, and if the lines [contained] by this sort of cone are extended, the portion of the luminous body's surface that they will envelop forms the part from which the light has radiated to the polished body according to two cones whose vertices lie at the [intermediate] point [just] imagined.

[3.90] And the light that propagates within these two cones radiates within and is contained by the first pair of cones, and the light propagates from the luminous source to the mirror along parallel lines, but these lines are included within the first pair of cones. Moreover, whatever lines the light follows as it radiates to the mirror, the lines of reflection maintain precisely the same relative disposition as the lines of incidence the light originally followed, so if the light radiates along parallel lines, it is reflected along parallel lines, and light falling within the extent of the polished body in the form of a cone is reflected in such a way as to form an equivalent cone.⁶⁴

[3.91] When light shines from a luminous body to a polished body through some hole, and if a point is imagined in the plane of the hole facing the luminous source, and from that point two cones are imagined, one with its base on the luminous body, the other on the polished body, light reaches the polished body through that point only from the base of a cone whose base is on the luminous body. Likewise, if a point is imagined in the plane of the hole facing the polished body, that point forming the vertex of two cones, one based on the mirror, the other on the luminous body, the light reaches the polished body through that point only from the base of a cone whose base is on the luminous body.

[3.92] So it is from the direction of the luminous body, which forms one common [base] of these two cones, that light shines toward the mirror, [which forms the other] common [base] of the two cones. Also, light reaches the mirror from the luminous source along parallel lines, and no matter what

lines it follows, it is reflected in the way described earlier. And all of the lines of reflection maintain an equivalent disposition with respect to the corresponding lines of the light's incidence, and in every reflection the light maintains the same precise form as it had in the polished body, and in what follows we will explain this in a clear manner.⁶⁵

[3.93] Now it has been established that the farther light extends from its source, the weaker it gets. It has also been established that concentrated light is more intense than dispersed light.⁶⁶ Thus, since light shines from any point on a luminous source to the surface of a mirror in the form of a cone, the farther it extends from that point, the weaker it gets for two reasons: because of its [increasing] distance from its source and because of its dispersal. Furthermore, when light is reflected from a point on a mirror, it is weakened in three ways: because of the reflection [itself], which weakens [the light]; because of its [increasing] distance from the point of reflection; and because of its dispersal.

[3.94] If, however, the light that is reflected from the mirror is concentrated at some point, it will be strengthened by that concentration, although it is weakened by the reflection [itself] as well as by its [increasing] distance [from its source].⁶⁷ Therefore, if the concentration of the light intensifies it as much as the fact of reflection and the distance [it lies from its source] weaken it, the concentrated reflected light will be as intense as it is at the mirror's surface. On the other hand, if the increase in intensity due to concentration is outweighed by the weakening due to the other two factors, the [reflected] light will be weaker [than it is at the mirror's surface], whereas if that increase outweighs [the weakening], the light will be more intense.⁶⁸

[3.95] Likewise, if the cone [of radiation] extends from the luminous surface [which forms its base] to some point on the mirror [which forms its vertex], the light radiating in the form of such a cone will be weakened by [increasing] distance but intensified by concentration. If, moreover, the [intensification caused by] concentration outweighs [the weakening caused by increasing] distance, the light concentrated at the point on the mirror will be more intense than a single [point of] light that radiates from the luminous source [to the mirror] along a single ray. I say "single," because a cone radiates from the luminous source to any point on the line chosen among [all those radiating from the given point on the luminous source], but such a cone is excluded from this consideration along with others of its kind.

[3.96] But if [the weakening due to] distance outweighs [the intensification due to] concentration, the light at the point [of incidence] on the polished surface will be weaker than the light taken along a single line of radiation, whereas if [the intensification due to] concentration outweighs [the

weakening due to] distance, the light [at the point of incidence] will be stronger [than the light taken along a single line of radiation].⁶⁹ Moreover, light that extends from the luminous source to the mirror along parallel lines will be weaker than that extending in the other way [i.e., along a cone with its base in the luminous object and its vertex at the mirror], for such light does not become concentrated on the mirror to counterbalance the weakening due to distance [from its source], and it proceeds along parallel lines after reflection. Hence it is weakened both by reflection [itself] and by [increasing] distance. But if it is concentrated upon reflection, that concentration will intensify it to the degree that it was intense [when it shone] upon the mirror, insofar as such concentration can balance out [the weakening due to both] reflection [itself] and [increasing] distance.⁷⁰

[3.97] Now, every line along which light radiates from a luminous body to a body facing it is a sensible line, not one without breadth, for light proceeds only from bodies, since it can subsist only in bodies. But in the least light that can be imagined there is some breadth, and [so] there is breadth in the line [or shaft] along which it radiates. And in the middle of that sensible shaft there is an imaginary line, and [all] the other [such] lines within that sensible shaft are parallel to it. So if the least [possible amount of visible] light is divided, neither part of it will constitute [actual, sensible] light; rather, both will be [effectively] extinguished and will [therefore] not be visible. On the other hand, if the least [possible amount of visible] light is doubled, or further multiplied according to equal increments, and if the resulting compound [light] is divided [equally], both of its portions will constitute [visible] light. But if [that compounded light] is divided unequally, one portion of it will constitute [effective] light, and the other [will be diminished] to the smallest [possible amount of visible light].⁷¹

[3.98] Now the smallest [possible amount of visible] light radiates upon the smallest portion of the [surface of a] body that light can occupy, and it radiates along an imaginary line centered within the sensible shaft [of radiation] whose edges are parallel to it. So the least quantum of light falls not upon an imaginary point on a [facing] body, but upon a sensible spot, and it is reflected along a sensible shaft that is the same breadth as the sensible shaft along which it reached [that body]. So if one conceives of an imaginary line in the center of the sensible shaft of reflection, it maintains the same disposition with regard to the point of reflection as the imaginary line in the center of the sensible shaft of incidence, and every line imagined in the sensible shaft of reflection maintains precisely the same disposition with respect to the imaginary line corresponding to it in the other sensible shaft [of incidence]. Thus, in the case of any light, the reflection of lines and of imaginary points follows this rule, even though light does not [actually]

proceed from such [points] along such [lines], and this is how the reflection of light will occur.⁷²

[3.99] Moreover, the reason that reflection occurs from polished rather than from rough bodies is because, as we have [already] pointed out, light approaches [any given] body with only the swiftest of motions, and when it reaches a polished body, that polished body causes it to rebound from it.⁷³ A rough body, on the other hand, cannot cause it to rebound because there are pores in the rough body into which the light enters; in polished bodies, though, it encounters no pores. But this rebound is not due to the [physical] resistance or hardness of the body, for we see reflection [occur] in water; on the contrary, this [kind of] repulsion is a function of polish, just as happens in nature when some heavy object falls from high above onto a hard stone [surface] and bounces back up, and the less hard the stone [surface] upon which it falls, the weaker the rebound of the falling object will be.⁷⁴ Furthermore, the falling object will invariably rebound in the opposite direction from that along which it [originally] traveled. The rebound that occurs in the case of a hard body does not, however, occur in the case of sand because of its softness.

[3.100] Moreover, [even] if there is some polish in the pores of a rough body, the light entering into those pores is nevertheless unreflected, and if it does happen to reflect, it is scattered and is not perceived by sight because of this scattering. By the same token, if the elevated portions of the rough body are polished, they will cause the reflected light to scatter, and on that account the reflection will be unnoticeable to sight. If, however, the height of these portions is slight, so that it is roughly the same as that of the lower portions, then the light reflecting from it will be perceived as if it came from a polished rather than from a rough body, even though [it will be perceived] less clearly [than light reflected from a perfectly polished surface].⁷⁵

[3.101] Now, the reason that the reflection of light will occur along a line that has an equivalent disposition with the line along which that same light will reach the mirror is because light moves extremely swiftly, so when it strikes the mirror it is not absorbed by the mirror; instead, its being trapped in that body is prevented. And since, on that account, it conserves the force and nature of its previous motion, it is reflected back in the direction along which it arrived and along lines that have the same [relative] disposition as the original lines [of incidence].

[3.102] We can observe the same thing in the case of natural as well as accidental motions.⁷⁶ If we drop a heavy spherical body perpendicular to a polished body from some height, we will see it reflected back along the perpendicular it followed in dropping. In the case of accidental motion, if the mirror is raised to the height of a man and is attached firmly to a wall,

and if a spherical body is attached to the point of an arrow, and the arrow is shot from a bow at the mirror in such a way that the arrow is at the same height as the mirror (and the arrow should be parallel to the horizon), it is obvious that the arrow reaches the mirror orthogonally, and you will see it rebound along the same orthogonal. If, on the other hand, the arrow flies along an oblique line with respect to the mirror, it will be seen to reflect not along the line according to which it arrived but along another one that is not parallel to the horizon, as was the case with the earlier path, but it will maintain the same disposition with respect to the mirror as that [original path] and with respect to the normal to the mirror.⁷⁷ Moreover, that the motion of light in reflection is due to the resistance of the polished body is evident from the fact that the more intense the repulsion or resistance [of that body], the more intense the reflection of the light will be.⁷⁸

[3.103] Here is why the motions of incidence and reflection turn out to be identical. When a heavy object falls orthogonally, the rebound from the polished body and the motion of the heavy body's incidence are perfectly opposed, so in this case there is motion only along the perpendicular. And the resistance occurs along the perpendicular, which is why the body is repelled along the perpendicular, so it rebounds orthogonally. When, however, that body impinges along an inclined path, the path of incidence lies between the normal that passes through the surface of that polished body and the line on its surface that is orthogonal to that normal.

[3.104] But if the motion [of the body] were to penetrate through the point [on the polished body's surface] that it strikes so as to find free passage, then the resulting line [of transit] would fall between the normal passing [through the surface of the polished body at the point of incidence] and the line on [the polished body's] surface that is orthogonal to that normal. Furthermore, it would maintain the same degree of orientation with respect to the normal passing [through the surface] as well as with respect to the other line that is orthogonal to that normal. For the measure of the disposition of this motion depends on the disposition [of its motion with respect] to the normal and the disposition [of its motion with respect] to the orthogonal [to that normal].

[3.105] However, since the resistance along the normal cannot balk the motion according to its measure along the line that passes orthogonally through the normal, because the impinging body does not penetrate [the reflecting surface], the polished body repels it according to the measure of its disposition with respect to the normal that is perpendicular to the orthogonal [that is parallel to the mirror's surface]. And since the motion of rebound has the same degree of orientation with respect to the orthogonal [to the normal] that it had with respect to that same orthogonal on the other

side, then the degree of its orientation with respect to the normal [after penetration] will be the same as it was before.⁷⁹

[3.106] But when the force of resistance that pushes a heavy, rebounding body upward is exhausted, that body, by its very nature, tends to drop to the center [of the universe]. Light, too, has the same intrinsic tendency to rebound, although it is not in its nature to rise or fall, so, in the case of reflection, it follows its original path until it reaches an obstacle that causes the [incident] motion to stop, and this is what causes reflection.⁸⁰

[3.107] It is also clear from [what we have established] above that colors radiate in tandem with light, so the reflection of color will be like that of light. And if you want to determine this fact observationally in the way described in chapter 2, you can do so by again using the [ringlike] apparatus. In attempting to observe such reflection accurately, you will not see [the relevant phenomena] clearly because of the weakness of color, for color is weakened by distance, by reflection [itself], and by the [restriction posed by the] hole through which it passes. That the [restriction posed by the] hole has a weakening effect is evident from the fact that light appears more intense after [passing through] a large hole than [after passing through] a small one. By the same token, when the holes [through which the color passes] are narrow, no color at all will be seen after reflection, or it will appear very faint. Nevertheless, if you wish to observe [the reflection of color] in the aforementioned apparatus, then you should make the mirror out of silver, for color appears very weak in an iron mirror, because it would be mingled with the reflected light, which is composed of the incident light and the dim light in the iron mirror, and the color of the iron mingled with the reflected color would weaken it.

[3.108] Once again, place the aforementioned apparatus in the room with only one window facing a white wall inside the room. Set the apparatus toward the room's window, which is narrow enough that [the space between] two of the holes in the apparatus can block it, [and let it be set up] so that the white wall facing it in the room can be viewed [in the inserted mirror] through either of the holes. Place a brightly colored body in front of the visible portion of the wall, and look at that portion of the wall through either hole in the apparatus. Accordingly, when the [colored] light enters through the holes in the apparatus, the color will be seen reflected through the hole corresponding to the one facing the colored body, [but it will be] invisible through the other hole. And this will happen with any [colored] body facing the hole, and what has been claimed about the reflection of light can be extended to the reflection of color. Moreover, the opening in the wall was as wide as the two holes in the apparatus exposed to it so that the light might shine more intensely upon the mirror, and the reflected color

is [thereby] more apparent. And because color is weakened when it shines directly through an opening and [is] likewise [weakened when] reflected, then, when it shines on a body facing the eye, it will be perceived as secondary, so if, after reflection, it shines on a white body that is exposed to the color at the opening, it may not be perceived by the eye because of the [resulting] weakening. When, however, the color at the second hole is exposed to the eye, it may be perceived because it will be seen as primary rather than secondary.⁸¹

[CHAPTER 4]

Part four, [showing] that the forms in bodies are perceived through reflection

[4.1] A number [of authorities] disagree about how the [visible] form is perceived in polished bodies. Accordingly, some of them [suppose] that rays emanate from the eye to the mirror, return from the mirror, and perceive the form of an object [seen in the mirror] upon its return.⁸² Others claim that the form of the object is impressed upon a facing mirror, so it is seen in the mirror the same way that natural forms of objects are perceived in objects.⁸³

[4.2] That in fact [the perception of images in mirrors] happens in another way [than this] is clear from the following. If someone sees himself moving in a given direction in some portion of a mirror, he will no longer see himself in the original place, but in a subsequent place, and this would not happen if his form were impressed on the original portion [of the mirror]. By the same token, if he moves to a third location, the place where his form appears [in the mirror] will shift, and his image will not be seen in the first or second location.

[4.3] Moreover, when some object is viewed [in a mirror] and the viewer draws away from it, it may happen that the object cannot be seen in that mirror, even though he may see the entire surface of that mirror, and this would certainly not be the case if the form were impressed on the mirror[']s surface] while the mirror remains in view and does not change its location, and while the object, too, remains stationary as its form is impressed on the mirror, just as before.

[4.4] In order to make it obvious that this is not how the form is perceived, block up half of the holes of the [ringlike] apparatus,⁸⁴ and place something with a letter on it at one of the blocked holes. If [the plane] mirror [inserted] in [its] panel is viewed through the hole corresponding to the hole where the letter lies, the letter will be perceived in the mirror, but it

will not be [perceived] through any other hole. If the form of the letter were impressed on the [surface of the] mirror, though, it could be perceived through any hole in the apparatus. Likewise, in the cylindrical mirrors, the situation of the letter will be perceived only through the corresponding hole. However, in the conical and spherical mirrors, both the situation and the size of the writing will be changed.⁸⁵

[4.5] Furthermore, if the cylindrical mirror is taken out [of the apparatus] and the panel [in which it is inserted] is placed [sideways] with its top and bottom edges standing vertical, the face of a person will appear frontal in it. If, however, the panel is placed upright or is placed at a considerable slant, [the viewer's face] will appear distorted.⁸⁶ So it is obvious that what is perceived will not be due to the form's being impressed on [the surface of the] mirror, since the visible object is not perceived in mirrors unless the eye is at the location of reflection. It is also obvious that the apparent distortion of the face is not due to the [actual] form of the object but to the disposition of the mirror [within which that form is seen].

[4.6] Moreover, if an object is viewed in a mirror and then drawn away, that object will be perceived [to lie] farther behind the mirror than before, and this will not happen if the form of the object lies on the surface of the mirror and is perceived there. Hence, reflection causes the form to be perceived [as it is] in a mirror.⁸⁷

[CHAPTER 5]

Part five, [showing] how forms are perceived in polished bodies

[5.1] It has just been shown in [the third] chapter above that, when a luminous colored body faces a mirror, light, along with color, radiates from any point on it to the entire surface of the mirror and is reflected along the appropriate lines of reflection. Thus, from any given point on a body facing the mirror, light, along with color, radiates to the mirror in the form of a continuous cone with its base on the surface of the mirror, and its form is reflected along lines whose disposition is equivalent to that of the lines of incidence, so the [cone of radiation] after reflection will be continuous with the cone of incidence. And if the form reaches the surface of an object [that faces the mirror as it radiates] along reflected lines, then, because those lines form a continuum, the entire form will fill the [facing] surface [of that object], leaving no area on it untouched. Accordingly, if the form of that body were to radiate to the mirror along these [same] lines (i.e., the [original] lines of reflection), and if it were to reach the base of the cone [on the mirror's

surface], then, since the lines forming this same cone [of incidence] are disposed in the same way as [subsequent] lines of reflection, the form is reflected along lines that form an [equivalent] cone, and the entire form will be concentrated on the point [previously] chosen [on the original luminous body].⁸⁸

[5.2] So, whenever the form of a given body reaches the mirror along some given lines, then, since those lines are disposed in the same way as the lines forming a cone that can be imagined projected to the mirror from a given point [on the other side], and since those lines correspond, the form [of the given body] will reach the given point according to that [imagined] cone. And if the eye lies at that given point, it will see the body that is represented by that form. And it has been established earlier that a form is apprehended in a mirror at a specific location.⁸⁹ Hence, the appropriate location and the natural way in which sight perceives [an object] in reflection are such that the lines [of radiation] along which the form reaches the mirror are disposed in the same way as the lines comprising the cone that is imagined to extend from the center of sight to the endpoints of each of those corresponding lines, and the reflected form cannot be perceived anywhere else but at this point.

[5.3] It is thus evident that the perception of forms [in reflection] occurs according to this arrangement of lines alone. It is also evident that light, along with color, radiates to the mirror from a luminous colored body and is [then] reflected [from it], and nothing but color and light radiate from that body. So it follows that the form of such a body is perceived on the basis of light and color alone, and since the form consisting of light and color radiates in such a way as to maintain the disposition described above, it is superfluous to claim, as many authorities have done, that rays pass from the eye to the mirror and then reflect to the place just described. So this [account of] how reflection occurs does not contradict, but accords with, the theory proposed by the geometers, for it preserves the disposition of the visual rays posited in the geometrical theory, and up to now this way [of accounting for reflection] has been evident only to me.⁹⁰

[5.4] However, since the form radiates from the luminous body to the mirror according to various dispositions along lines that are imagined [to extend] from each point on the body to the entire surface of the mirror, the same form will be reflected according to various cones with their vertices at various points and their bases on the surface of the mirror, [and they will be reflected in such a way as] to maintain a [corresponding] disposition with the lines along which the form radiates. From this it follows that at any given instant, with the mirror fixed in place, the form of a body is perceived from various viewpoints where the vertices of the cones of reflection lie. By

the same token, when a given center of sight passes over the vertices of such cones while the mirror remains stationary, the same form will appear at different locations in the mirror. But when the same form is perceived at different places in the mirror, the lines of sight are directed to different spots on the mirror, for they cannot grasp the forms of different points on the body identically at the same spot on the mirror.

[5.5] It has already been asserted that light radiates from any point on a [luminous] body to any point on a mirror, so at any point on the body lies the vertex of a cone whose base is the surface of the mirror. Also, any point on the surface of the mirror constitutes the vertex of a cone whose base is the surface of the body. Thus, the entire form of the body will be [radiated] to each point on the mirror along lines that go in different directions, and they cannot possibly intersect [before that point]. In addition, the form reaching from the [luminous] body to any point on the mirror in the form of a cone will be reflected in the form of a cone. And the same process by which the form reaches the mirror can be multiplied indefinitely, for the entire form overlaps with each part and at each point [on the mirror]. And there is no interruption in such forms; instead, there is an absolute continuity in reflection. Still, since the entire form does not fall at disparate spots on the mirror, it is directed with the same disposition to different spots where the viewer perceives it.⁹¹

[5.6] Thus, if the shape of the mirror is the same as the shape of the object [seen in it], the complement of the form and shape of the body will be [seen] in the mirror. For in the case of a mirror that is the same shape as the object, the form of the first point [on that object] is directed to the first point on the mirror, the second to the second, and so forth among all the corresponding points. Hence, the shape of the entire object will be [seen] on the mirror's surface, which does not happen in the case of a mirror with a different shape.⁹² Likewise, if some portion of the mirror is taken, and if it is the same shape as the object [seen in it], the complement of the shape of the object will be [seen] in it. Moreover, since such parts on the mirror are infinite, the forms of the body that are reflected will be infinite, but they will radiate to different points where the viewer perceives the form.

[5.7] Thus, since the form is perceived according to this arrangement of lines, the form radiating from the body will not be impressed on the surface of the mirror. And this is the way reflection occurs for all mirrors, but most evidently in plane mirrors; in the other [sorts of mirror], however, some variation occurs because of visual error, which occurs in the way previously mentioned.⁹³ So, according to the way just described, any [given] center of sight perceives only one of the object's points at one point on the mirror, and the same point on the object is not perceived by two centers of sight at the same point on the mirror.

[5.8] Furthermore, suppose that a mirror faces the eye, and imagine a cone [extending] from the center of sight to the surface of the mirror. Let a point be taken on the base of this cone, and imagine a line [extending] within this cone from the center of sight to that point. Then, given that an infinite number of lines can be extended from that point, since each of them maintains an equivalent disposition with respect to its cone's edge and forms an equal angle with the normal, and since this holds for every point taken on the mirror, it is clear that reflection can occur from every point on the mirror. I say, therefore, that among [each of] the lines extended from the given point there is a line that is equivalently disposed with respect to the edge of the cone, and it forms an equal angle with the normal to that point. And this [line] forms the edge of a cone imagined to extend from that point to the surface of the [visible] object, and whatever lies at the endpoint of that line, since [its form] reaches the given point [on the mirror] along that line, [that form] will be reflected to the center of sight along the edge of the cone just described. Furthermore, this edge of the cone will lie in the same plane as the line projected from that point, and that plane is perpendicular to the plane tangent to the mirror at that point. And I say [that] this [is so] when the edge of the cone is oblique to that point. If, in fact, the [line along the] edge of the cone extending from the center of sight falls orthogonally to the plane tangent to the surface of the mirror at the given point, it will be reflected back onto itself and will return to the source of its propagation in the eye.

[5.9] What we have claimed is obvious in the case of a plane mirror, for at whatever point on the [mirror's] plane surface the ray from that point [on the given object] falls, a normal can be erected to that surface. And from the center of sight a line can be imagined to fall orthogonally to the surface continuous with the aforementioned surface, or to the surface itself. And these two perpendiculars will lie in the same plane, because they are parallel, and the line extending in that plane surface from the endpoint of one to the endpoint of the other will form an acute angle with both and will lie in the same plane with both. So the ray projected according to that line will form an acute angle with the normal to the mirror [at the point of reflection], and likewise [it will form an acute angle] with the perpendicular drawn from the center of sight.⁹⁴ Then imagine a line drawn on the other side of the plane surface passing orthogonally through the endpoints of the perpendiculars. On that other side it will form a right angle with the normal to the mirror, so from that right angle an acute angle can be cut that is equal to the acute angle the ray forms with that same normal. These two angles [will lie] in the same plane, so the incident and reflected rays lie in the same plane as the two aforementioned perpendiculars. Moreover, if some other

point [on the visible object] is viewed [in the mirror], the [two] rays will be equivalently disposed with respect to the perpendiculars, one of those rays extending from the point viewed, the other from the center of sight.

[5.10] Hence, in every plane of reflection four points occur in a related group—i.e., the center of sight, the point [on the visible object] that is perceived, the endpoint of the perpendicular [dropped] from the center of sight [to the mirror's surface], and the point of reflection.⁹⁵ And all the planes of reflection [for the given point of reflection] intersect along the normal imagined to extend to the point of reflection, and this line is common to all the planes of reflection. And since the same thing happens when any point on the plane surface is viewed, there will also be an identical reflection from all points, and it will occur in the same way.

[5.11] Furthermore, what we have claimed will be clear in the case of [convex] spherical mirrors. If a [convex] spherical mirror faces the eye—and such a facing situation means that the eye does not lie on the surface of the spherical mirror itself or on the spherical surface continuous [with that surface]⁹⁶—and if one looks into this mirror, the portion of it that is perceived will be the portion of the sphere defined by the circle cut off by a ray tangent to the surface of the sphere that rotates [axially] about that circle of tangency until it returns to the original point where the rotation began. And if one imagines planes intersecting along the diameter that is imagined to extend [through the sphere] from the pole of the aforesaid circle, any of the arcs upon the surface of the sphere that are imagined to be the common sections formed by these planes on the surface of the sphere between the pole and the [defining] circle will be less than a quarter of the great circle [of the sphere], because the line from the center of the sphere to the endpoint of [any] ray drawn tangent to the sphere—that is, to the aforementioned [defining] circle—forms a right angle with the ray on account of its tangency. Hence, that tangent ray forms an acute angle with the radius drawn from the circle's pole, and this angle corresponds to the arc between the circle's pole and the [defining] circle itself, so every one of these arcs is less than a quarter of the [great] circle.⁹⁷

[5.12] I say, then, that reflection can occur from any point on this portion [of the sphere], for, if any point on it is taken, the diameter of the sphere imagined [to extend] from this point will be perpendicular to a plane tangent to the sphere at that point. The proof of this claim is as follows. If two planes are imagined to cut the sphere along the diameter imagined [to extend] through the given point, the common sections of the sphere's surface and these planes form [great] circles on the sphere that pass through the given point. And if two lines are imagined tangent to these circles at the given point, the diameter will be orthogonal to both of the lines, so [it is

orthogonal] to the plane in which they lie. And when a ray is incident to the given point, it will lie in the same plane as the sphere's diameter, whose endpoint is the given point, and [it also lies in the same plane as] the line imagined [to extend] from the center of sight to the center of the sphere, which of course passes through the circle's pole, and this radial line also falls orthogonally to the surface of the sphere. So a triangle will be formed by these three lines, and the ray that is incident to the given point forms an acute angle with the diameter of the sphere [extended] beyond [its surface], for, since this ray lies above the ray that is tangent to the sphere [at the given point], it will cut the sphere when it is imagined to extend [into the sphere]. And the plane tangent to the sphere at the given point will lie below this ray, and it will extend between the sphere and the visual axis—i.e., the line that is imagined to pass from the center of sight to the center of the sphere through the pole of the [defining] circle [formed by the rotating tangent].

[5.13] Therefore, since the diameter of the sphere is perpendicular to the plane that is tangent [to the sphere] at the [given] point, it will form an angle greater than a right angle with the [incident] ray inside the circle at the point of incidence, so outside the circle it will form with it an angle less than a right angle. And when [this diameter is] extended beyond [the sphere's surface], it will be orthogonal to the plane tangent [to that surface at the given point], so within the right angle that it will form with that plane on the other side of the ray an acute angle can be cut off equal to that [original acute angle] formed by a ray with that diameter. And the three lines forming these two angles will lie in the same plane, so a line can be drawn from the point on that portion [of the surface] that lies in the same plane as the ray that is incident to that point, as well as in the same plane as the normal to the plane tangent to the surface at that point, and it can form an equal angle with that normal. And the form of a point propagated to the mirror's surface along that ray will reach that line. Thus, it is equivalently disposed with respect to the line along which it can be reflected, and the plane in which these lines lie is orthogonal to the surface of the sphere at the point of tangency, and this should be understood to hold for every point on the [visible] portion [of the spherical mirror].⁹⁸

[5.14] Therefore, the center of sight, the center of the sphere, a point of reflection, and a point [whose form is] reflected will lie in every plane of reflection, and all such planes will intersect along the line that extends from the center of sight to the center of the sphere. And the common section for each plane of reflection and the surface of the sphere will be a [great] circle on that sphere, and all [these] circles will intersect at the point on the [surface of the] sphere where the visual axis falls, and this [point] lies on the

pole of the circular section [of visibility defined by the tangent rays].⁹⁹ Moreover, when the ray strikes the mirror along the perpendicular to the surface where the plane tangent to the sphere touches the sphere's surface at the point where the ray strikes—and this ray constitutes the visual axis [that passes] along the pole of the circular segment to the center of the sphere—the reflection will take place along the same radial line to the eye according to the source of the ray's propagation.

[5.15] Now, what we have said [above] will also be clear in the case of [convex] cylindrical mirrors. Place a cylindrical mirror that is polished on the outer [convex] surface directly facing the eye—and such a facing situation means that the eye does not lie on the surface of the cylindrical mirror itself or on the continuation of that surface—and we shall imagine a plane [passing] through the center of sight that cuts the surface of the cylinder along a circle that is parallel to the bases of the cylinder. Then, in this plane take two lines that are tangent to the circle of intersection at two opposite points. From each of those points extend a line along the length of the cylinder, and imagine two planes that contain these two lines [extending] along the [cylinder's] length as well as the two lines that are drawn from the center of sight to the circle of intersection and tangent to it. I say that these [two] planes are tangent to the cylinder.

[5.16] For if it is claimed that either of them cuts it, then it is obvious that an intersection will occur upon a line that extends along the length of the cylinder where the plane falls on its surface, and likewise an intersection will occur along a line of longitude on the opposite side. And the circle of intersection passes through these two lines of longitude. In addition, since the line tangent to the circle of intersection lies in some plane, it cuts the cylinder according to some [given] lines along the length that are parallel to one another, and if it intersects one of them it will intersect the other, both cuts being at equal angles. Therefore, if it passes through the point where the circle of intersection touches the first line along the [cylinder's] length, it will also pass through the point where the other line along its length touches that circle. Accordingly, it cuts the circle, so it will not be tangent to it, and this is counter to what we supposed. It is clear, then, that these two planes are tangent to the mirror and that whatever lies between them upon the surface of the mirror is visible to sight.¹⁰⁰

[5.17] Furthermore, since these two planes meet at the center of sight, they will intersect, and their common section will pass through the center of sight, and it is parallel to the axis of the cylinder, because the cylinder's axis is perpendicular to the circle of intersection. The lines along the length of the cylinder are also perpendicular to that circle, and the planes tangent to the cylinder along these lines are perpendicular to that same circle. So

they are perpendicular to the plane cutting the cylinder along that circle, and the common section of these planes is perpendicular to that same plane, so they are parallel to the cylinder's axis.¹⁰¹

[5.18] I say, then, that whatever point is taken on the visible portion of the mirror, the line extended from the center of sight to that point will cut the mirror. For if the line extended from that point along the length of the cylinder is imagined, it will pass through the circle of intersection and will touch it at a point where a line drawn from the center of sight will intersect the portion of the mirror that lies between the lines tangent to this circle. Also, the plane passing through the center [of sight] that contains this line will cut the cylinder. Therefore, since this line lies in the same plane as the line drawn from the center of sight to the point chosen [on the mirror's surface], it will intersect the mirror, and so, any line that is imagined [to extend] from the center of sight to the [visible] portion of the mirror cuts the mirror. By the same token, any line that is imagined to extend from the common section [of the two tangent planes passing] through the center of sight to that portion [of the mirror] cuts the mirror, so whatever plane is tangent to the mirror along any line on the visible portion cuts the planes that are tangent at that portion's edges. And not one of all the planes tangent to that portion passes through the center of sight; instead, it will extend between the eye and the mirror.

[5.19] Accordingly, I say that light can be reflected from any point on this portion [of the mirror]. For, given such a point, let a circle parallel to the bases of the cylinder be drawn through it. Hence, if the plane passing through the center of sight cuts the cylinder along a plane parallel to the bases, it should cut the cylinder along that circle, and the line extended from the center of sight to the center of the circle should pass through the given point. The form of that point will be reflected back along the same line to its origin-point, because that line is the visual axis [which is] perpendicular to the axis of the cylinder. Furthermore, if any point through which the [visual] axis passes orthogonally to the axis of the cylinder is taken, the reflection of that point will take place along that very [visual] axis.

[5.20] But if the chosen point lies outside this axis, then any line extended to that point from the center of the circle drawn parallel to the bases [of the cylinder]—that line also being extended to the plane tangent to the line along the length of the cylinder passing through that point—will be orthogonal to the [cylinder's] axis, so it is orthogonal to the line along the length [of the cylinder] passing through that point. And since the center of sight lies above the plane tangent [to the cylinder] at that point, the line drawn from the center of sight to the chosen point will form an acute angle with the normal extended to that point from the center of the circle. This is

the case outside [the cylinder], so [the angle formed] inside [the cylinder] is obtuse. And from the right angle that the normal forms with the line on the plane that is tangent to the circle [of intersection] an acute angle the same size [as the aforementioned angle] can be cut. And that normal lies in the same plane as the center of sight, so it lies in the same plane as the line drawn from the center [of sight] to the [chosen] point. So the reflected ray will lie in the same plane [as this line], and this plane will be orthogonal to the plane tangent to the mirror at that [chosen] point, for the normal falls orthogonally to this plane. And this is how the plane of reflection will be defined.¹⁰²

[5.21] Furthermore, the common sections of the planes of reflection and the surface of the cylinder vary among each other, for, when reflection occurs [back] along the same ray [along which the form reaches the mirror], that same ray will fall orthogonally to the axis [of the mirror]. And the common section of the cylinder's surface and the plane of reflection will be a straight line—i.e., an edge of the cylinder—since the diameter of the cylinder lies in the plane of reflection. And this is clear because a cylinder is generated by the rotation of a plane with parallel sides about one of those sides, which remains stationary [to form the axis of rotation]. Accordingly, the common section of a plane that cuts the cylinder through its axis—i.e., the stationary side [of the generating surface]—and the surface of the cylinder will be the side that moves [in the process of generation]. And I claim that only one among all the planes of reflection forms a rectilinear common section with the surface of the cylinder, for only one plane, and no more, can be imagined to contain the axis of the cylinder and the center of sight.

[5.22] On the other hand, if the plane of reflection is parallel to the bases of the cylinder, the common section will be a circle, and this is the only plane that can form a circular common section with the surface of the cylinder, for in the case of every reflection, the normal to the plane that is tangent to the point of reflection forms the diameter of a circle parallel to the bases of the cylinder. And on the surface of the cylinder there can be but one circle that is parallel to the bases and that lies in the same plane as the center of sight. All the remaining planes of reflection cut the [surface of the] cylinder and the cylinder's axis, for the normal dropped to the point of reflection cuts the cylinder's axis, but the common sections of these planes and the surface of the cylinder form the sections that geometers attribute to cylinders and cones [i.e., ellipses].¹⁰³

[5.23] When the common section of the surface of the cylinder and the plane of reflection is a straight line [along the length of the cylinder], then, to whatever point on that line one directs his sight, reflection occurs in the same plane as the axis, for there is only one plane tangent to the cylinder

along that line of longitude. So, given some point on this line, the perpendicular extended from it to the axis will lie in the same plane as the axis, and this extended line is perpendicular to the plane that is tangent to the surface of the cylinder [along that line of longitude]. But the center of sight lies in a plane orthogonal [to the plane of tangency], and so it lies in the same plane [as just described], and the axis of the cylinder and the common section lie in that plane, and there is only one plane orthogonal to that surface along the same common section, so all reflections that take place from points on this line lie in the same plane of reflection.

[5.24] However, when the common section of the plane of reflection and the [surface of the] cylinder forms a circle, then, if any point on that circle is viewed, reflection will occur in one and the same plane, because any perpendicular extended from the point that is viewed will form a diameter of this circle, so it lies in the plane of that circle, as does the center of sight. And the plane of this circle is orthogonal to the plane tangent to the circle at whatever point is chosen on it, so in this single plane reflection will occur from whatever point [is chosen] on the aforementioned circle. If any other common section is taken, however, reflection will occur in the same plane of reflection at only one point on this line [of section], for the perpendicular extended from the point of reflection is orthogonal to the line along the length of the cylinder that passes through that point, so it is also perpendicular to the axis. But that perpendicular forms the diameter of a circle parallel to the bases of the cylinder. So the plane of reflection and that circle intersect, and their common section forms a diameter of that circle, and it is perpendicular to [the cylinder's axis], and the plane of reflection cuts [that axis] but is oblique to it. But in a plane that is oblique to a given line, only one line can be imagined orthogonal to that line. If reflection were to take place from two points within the same plane of reflection, two lines within that plane would be perpendicular to the axis, and that cannot be, since the plane is oblique to the axis.¹⁰⁴

[5.25] Moreover, the normal through the point of reflection falls on a circle that is parallel to the bases of the cylinder, and [it falls on] a point on the axis that is common to the circle and the plane of reflection. If, therefore, reflection were to occur from any other point on the common section [of the cylinder and the plane of reflection] within the same plane, there would be another normal dropped from the other point, and it would form the diameter of another circle on the cylinder parallel to the first one, and it would fall on a point on the axis where the plane of reflection does not fall. And so it must be understood that in every [oblique] plane of reflection, reflection occurs from only one point on the common section [of the cylinder and the plane of reflection] in the same plane with respect to the same

center of sight. Hence, with respect to both eyes [reflection] can take place from the two endpoints of the circle's diameter, which constitutes the normal. Yet with respect to one eye this does not occur, because those two points cannot be perceived at the same time by the same eye, for [with one eye] it is necessarily the case that less than half of the cylinder is seen.¹⁰⁵

[5.26] From the foregoing it is evident that the perpendicular imagined to extend to the point of reflection from outside [the cylinder] forms the diameter of the circle as it passes inside [the cylinder], for, if such were not the case, then, since it is agreed that the diameter of the circle, which passes to that point, is perpendicular to the plane tangent to the [surface of the] cylinder at that point, and [since it is agreed that it is] also perpendicular [to that plane] on the outside [surface of the cylinder], there will be continuity between these perpendiculars, and they will form one line. For if the diameter is not perpendicular to that plane [of tangency] when it is extended beyond [the cylinder's surface], it will follow that two perpendiculars could be drawn to the same point on a [given] plane. Hence, it is clear that in every plane of reflections four points occur in a related group: the center of sight, the point on the axis where the normal falls, the point that is seen on the mirror [i.e., the point of reflection], and the point from which the form of the [visible] object radiates.

[5.27] In the case of conical mirrors that are perpendicular to their bases [i.e., formed from right cones] and that are polished on their outer [convex] surfaces, the eye is considered to face them if it does not lie on the surface of the mirror or on the continuation of that surface, and how much of [the surface of the] mirror is seen will depend on how the eye is disposed with respect to the mirror.

[5.28] Accordingly, if the ray from the center of sight to the endpoint of the cone's axis—i.e., [the ray that is] understood [to extend to] the vertex—forms an acute angle with the axis on the [visible] side of the cone, we will imagine a plane [passing] through the center of sight and cutting the cone's surface along a circle that is parallel to the base of the cone. And we will imagine two lines from the center of sight that are tangent to that circle at opposite points, and from those points we will draw lines along the length of the cone. Hence, the plane containing either of these lines along the length [of the cone] as well as the [corresponding line that is] tangent to the circle will be tangent to the cone, for if it were to cut the cone, it would touch some other point besides the point of tangency on the circle. On that [other] point draw a line along the length of the cone, and that point and the vertex of the cone lie in this plane, so that line will lie in this plane and will pass through some point on the circle. That point, therefore, lies in this plane and on the circle, so it lies on a line that is common to the circle and to the

surface [of the cone]. But it is [also] tangent to the circle, so this tangent passes through two points on the circle that it touches, which is impossible.¹⁰⁶ It follows therefore that this plane is tangent to the [surface of the] cone.

[5.29] And it is invariably the case that any plane of reflection in which the line that is tangent to some point on the cone and the line of longitude that passes through that point intersect is tangent to the cone along the line of longitude. Hence, we have two planes passing through the center of the eye and tangent to the [surface of the] cone, and in this situation the portion of the cone between them is visible to sight, and it constitutes less than half the cone, because the lines that are tangent to the circle encompass less than half of it.¹⁰⁷

[5.30] If, on the other hand, the line drawn from the center [of sight] to the vertex of the cone forms a right angle with the axis, then imagine a circle that cuts the cone so as to be parallel to the base. The common section of this circle and the plane that contains the axis of the cone and the center of sight will be [a line] perpendicular to the axis of the cone, because the axis is perpendicular to the plane of the circle. To this common section draw a diameter through the center of the circle perpendicular to this line [that forms the common section], and from the endpoints of this diameter draw two [lines] tangent to the circle, and also draw two lines to the vertex of the cone. The two planes in which these latter two lines lie, along with the lines tangent [to the circle], are tangent to the cone in the way we described before. And since the common section of the circle and the plane in which the center of sight and the cone's axis lie is parallel to the line drawn from the center of sight to the endpoint of the axis, and since the lines drawn tangent to the circle at the aforementioned points are parallel to this common section, those tangents will be parallel to the line drawn from the center of sight to the endpoint of the axis, so they will [each] lie in the same plane as that line. Hence, both planes that are tangent to the circle pass through the center of sight, and the common section of those planes is the line drawn from the center of sight to the endpoint of the axis. So whatever lies on the cone between those planes is visible to sight, and it constitutes half the cone, because half the circle lies between those lines of tangency. And so it is evident that in this situation half the conical mirror is visible.¹⁰⁸

[5.31] However, if the line extended from the center of sight to the endpoint of the cone's axis forms an obtuse angle with the axis from a point of view above the cone, and if a circle is formed to cut the cone parallel to the base, the [line forming the] common section of this circle and the plane containing the center of sight and the axis is perpendicular to the cone's axis. When this common section is extended beyond [the back edge of the cone],

it will intersect the line drawn from the center of sight to the endpoint of the axis according to an acute angle, [and] an acute angle is [also] formed by this line [of sight] with the axis below [the point of intersection]. From the point of intersection of these lines, draw two lines tangent to the circle at two opposite points, and draw the lines [of longitude] from these points to the vertex of the cone. The planes containing the lines tangent [to the circle] and these lines along the length [of the cone] are tangent to the [surface of the] cone, and in both of these planes two points on the line extended from the center of sight to the endpoint of the axis lie—i.e., the endpoint of the axis and the endpoint of the perpendicular where that line [drawn from the center of sight to the endpoint of the axis] and the perpendicular intersect—so that line lies in both planes. Therefore, both planes pass through the center of sight, and on the inner, lower side [of the cone] these planes encompass a portion of the cone that is less than half, because the lines tangent to the circle encompass a portion [of the circle] that is less [than half]. Hence, from [the point of view] above [the cone], more than half [the cone] lies between the planes that are tangent to [the surface of the] cone, and that is what is visible to the eye, so in this situation the eye perceives more than half the [surface of the] cone.¹⁰⁹

[5.32] On the other hand, if the line extended from the center of sight to the endpoint of the axis falls on an edge of the cone so that this line and the edge form a continuous line, I say that none of the cone will be hidden from sight except for the given line imagined [to extend along the edge], for every plane that contains the line extended from the center of sight to the endpoint of the axis and a line extended along the length of the cone's edge cuts the cone except where it touches the cone along the edge that forms part of the line [drawn from the center of sight to the axis]. And [of all the lines] imagined to lie on the surface of the cone in this situation, this edge-line alone passes [unseen] by the center of sight.

[5.33] The truth of this point is evident from the fact that, whatever point is chosen on the surface of the cone, if a line is drawn to it from the center of sight, and if from that same point a line is drawn along the length of the cone to the endpoint of the axis, these two lines will form a triangle with the line drawn [from the center of sight] that coincides with the edge [of the cone]. And this triangle lies in a plane that is imagined to pass through the center [of sight] so as to cut the cone, and among [all] the lines lying in this plane only two fall on the surface of the cone—i.e., the line [that coincides with the line along] the edge and the line along the length [that extends] from the chosen point to the vertex of the cone. But the line extended from the center [of sight] to the chosen point cuts the line along the length [of the cone] at the chosen point of reflection, and it cuts the line that coincides

with the edge of the cone at the center of sight, so none of these lines [of longitude] will intersect that line [along the edge of the cone] outside the center of sight. Therefore, since no other point can be chosen to which the line from the center of sight reaches and through which it passes, this point is not blocked by any other point. And so it is visible to the eye, since there is no solid object lying between it and the eye. The same can be demonstrated for any [other] point on the surface of the cone.¹¹⁰

[5.34] Moreover, if the line [extended] from the center of sight to the endpoint of the axis enters the cone, I say that no point on the entire surface of the cone is hidden from the eye. For, given some point on the surface of the cone, imagine a line drawn to it from the center [of sight], and imagine another drawn from it to the vertex of the cone. These two lines enclose a triangular surface with the line extending from the center of sight to the endpoint of the axis that enters the cone, and this triangle lies in a plane that cuts the cone's surface. Since every plane containing a line that enters the cone cuts the cone, then the line extended from the center [of sight] to the chosen point cuts at that point the line along the length of the cone that extends from it to the cone's vertex. And among [all] the lines lying in the plane that contains these two lines, there are only two that lie on the cone's surface—i.e., the one line that is extended along the length of the cone from the [chosen] point to the [cone's] vertex and the other line that is [directly] opposite, which cuts the angle formed by the first line with the line that enters the cone. Thus, when it is extended beyond the cone, the line on the opposite side cuts the line extended from the center [of sight] to the chosen point; hence, within the plane [formed by these lines] the latter line [extending from the center of sight to the chosen point] cuts only two lines that lie on the surface of the cone, [cutting] one beyond the cone and the other at the chosen point, so if it is extended to infinity it will intersect none of the other lines [along the length of the cone]. According to the earlier account, then, the chosen point is not blocked from sight.

[5.35] Therefore, in this situation none of the planes tangent to the cone will pass through the center of sight; rather, each of them will intersect the line of sight that enters the cone through the endpoint of the cone's axis between the center of sight and the cone, and the [point of intersection] lies at the endpoint of the axis. But when the line of sight coincides with a line along the length of the cone, none of the planes tangent to the cone will reach the center of sight except the one that is tangent to the cone along the aforementioned line. So all the tangent planes will cut that line between the center of sight and the cone.¹¹¹

[5.36] Likewise, in the case when two planes that are tangent to the cone pass through the center of sight, any plane that is tangent to the cone in the

visible portion of the cone that lies between the two tangent planes misses the center of sight. No matter what point of that portion the line of sight reaches, that line will cut the cone, because that portion lies between the two lines of sight that are tangent [to the cone]. And the plane containing this line of sight [i.e., the one reaching the given point on the cone's surface] and the line along the cone's length will cut the cone, and this will be the visual plane for whatever place on the cone's surface it touches within that portion, and so [that place on the cone's surface] is also visible.¹¹²

[5.37] I maintain, therefore, that in any [such] situation reflection can occur from any given point [on the visible portion of the mirror]. Choose a point, and imagine a circle passing through that point and parallel to the base of the cone. The diameter originating at that point on this circle will be perpendicular to the axis, since the axis is perpendicular to the plane of the circle, so the line along the length [of the cone] extended from the [chosen] point to the vertex of the cone forms an acute angle with the diameter as well as with the endpoint of the axis within the same plane [i.e., the plane containing the diameter, the axis, and the line of longitude]. Let a line of sight fall to the [chosen] point within the plane containing the line along the length [of the cone] and the axis, and in that plane draw a normal to the line along the length [of the cone] at that point. This normal will of course intersect the axis, and a triangle will be formed by this line with the axis and the line along the length [of the cone]. Imagine a line tangent [to the circle] at that [chosen] point, and imagine another diameter in the circle we drew that is orthogonal to the original one and orthogonal to the axis as well, so it will be orthogonal to the plane containing the axis and the first diameter. This second diameter is parallel to the line of tangency [just drawn], because that line of tangency is perpendicular to the first diameter. Hence, the line of tangency is perpendicular to the plane containing the axis and the first diameter, so it will be orthogonal to the normal that we just drew. And so that normal falls orthogonally to the plane tangent to the cone at the point we chose.

[5.38] Accordingly, if the line of sight that falls upon the chosen point passes to it along the normal, it will be orthogonal to the plane tangent to the cone at the chosen point, and the [visible] form will be reflected back along that same line of sight. On the other hand, if the line of sight does not pass along the normal, it will form an acute angle with that normal at the chosen point. And in the plane of this line of sight another line can be drawn from that point to form an angle equal to that which it forms with the normal, because that normal is orthogonal to the plane of tangency. Moreover, any line that falls orthogonally to the plane of tangency at the chosen point passes to the axis. And if a perpendicular is drawn to that

plane from the axis, the inner and outer segments of that perpendicular will form a single line. For, if the interior segment of the perpendicular is not also perpendicular to the plane when it is extended outside [the cone], it will follow that from the same point on that [same] plane two perpendiculars can be erected on the same side [which is impossible].

[5.39] So it is evident that reflection can take place at equal angles from any point that is viewed on the surface of the cone. And when the [visible] form reaches the line of reflection, it will arrive at the mirror along that [original] line and will be reflected to the eye along the other, and these two lines lie in the same plane, which is orthogonal to the plane that is tangent to the cone at the point of reflection. This is the plane of reflection, and in it four points are invariably grouped together: i.e., the center of sight, the point that is seen, the point of reflection, and the endpoint of the normal.¹¹³

[5.40] The common sections of the planes of reflection and the surface of the cone [can] vary, however. For when the line of sight coincides with the axis of the cone—i.e., when the entire axis of the cone and the normal [to the cone's surface] that passes to the axis lie in every plane of reflection—then in this case the common section of each plane of reflection and the surface of the cone will be a line along the length [of the cone]. For every plane that contains the entire axis [of the cone] has this [particular] common section with the surface of the cone.

[5.41] But in every other case only one line along the length of the cone will form the common section—i.e., the common section lying in the plane that contains the center of sight and the axis. And because the center of sight does not lie in a direct line with the axis, there will be only one such plane [of section], and every other common section will be a conic section, not a circle. For if it were a circle, the plane of that circle would lie in the plane of reflection, and since the [cone's] axis is orthogonal to that circle (because any circle [of section] on the cone is parallel to its base), the edges of the cone will be inclined to the circle, as well as to the plane of reflection. In that plane [of the circle], therefore, a normal cannot be drawn to the line along the length of the cone. But the normal extended to the plane that is tangent [to the cone] at the point of reflection lies in the plane of reflection, and it is perpendicular to the line along the length [of the cone], because any plane that is tangent to the cone is tangent [to it] upon a line along its length.

[5.42] It therefore follows [that it is] impossible [for the plane of reflection to form a circular section with the cone], so all the other common sections of [the plane of] reflection [and the surface of the cone] are conic sections, and when the common section is a line along the length [of the cone], reflection can occur from any point on that line in the same plane as reflec-

tion from any other point of reflection. For the perpendicular extended from any point on that line will meet the axis; and the center of sight, the point of reflection, and the point on the axis [where the normal intersects it] will lie in the plane of reflection; and the line along the length [of the cone] and the plane within which this reflection occurs is the one formed by the line along the length [of the cone] and the axis, so reflection takes place from any point in this plane.¹¹⁴

[5.43] If, however, the common section is not a line along the length [of the cone], I say that reflection may take place from either one point, or at most two points, on the common section within the same plane. For, if a normal is extended from the point of reflection, it will reach the axis and will fall on one of its points. When a circle is imagined [projected] through the point of reflection, that circle will cut the axis orthogonally, and since the perpendicular [in the plane of the circle] cuts the axis [in a plane] parallel to the [cone's] base, the normal [to the cone's edge] will be inclined to the [plane of the] circle. And when it is rotated about [this circle] that normal will always be equal[ly disposed with respect to the circle], so it will form a cone whose base is the circle and whose vertex is the point at which the normal intersects the axis. Therefore, the plane of reflection will either be tangent to, or will cut, this [second] cone.

[5.44] If it is tangent to it, I say that reflection can occur in the same plane from only one given point of reflection. It is evident that the plane of reflection will be tangent to this [second] cone along the normal, which is an orthogonal line in the plane of reflection, and if lines are drawn from the vertex of the entire [second] cone to the common section of the plane of reflection and the larger, original cone, they will fall on the circle that forms the base of the imagined cone, but only one falls on that common section at the point of reflection. Therefore, if reflection were to take place from another point on this common section, the line imagined to extend from that point to the [second cone's] vertex would be perpendicular to the line along the length passing through that point on the [original] cone. But the line [extending] from the vertex of the imagined cone to the point on the circle through which the line along the length of the [mirror's] cone passes is certainly perpendicular to this latter line, so any other line forms an acute angle, not a right angle, with it.¹¹⁵

[5.45] On the other hand, if the plane of reflection cuts the imagined cone, it will cut the circle at its base in two points. I say that these are the only points on the entire common section [formed by the cutting plane] from which reflection can take place in the same plane, for the line extended from either of these points to the vertex of the imagined cone is perpendicular to the line along the length [of the mirror's cone] at the point through

which it passes. From any other point on the section [formed by the cutting plane], the line extended to the vertex of this [imagined] cone will form an acute angle with the line along the length [of the mirror's cone] through which it passes, since the normal to the same line along the length [of the mirror's cone] forms a right angle in the circle. So the lines drawn from the vertex of the imagined cone to points on the section lying between the vertex of the mirror and the circle will form obtuse angles with the lines along the cone's length on the side opposite the vertex of the entire cone [of the mirror]. And the lines drawn to the points lying between the circle and the base of the mirror form acute angles with the line along the length [of the cone] on the side opposite the vertex while forming obtuse angles on the side opposite the base.¹¹⁶

[5.46] In the case of spherical concave mirrors, if the center of sight is located within the hollow of the mirror, the entire surface of the mirror will be visible to it. However, if the eye lies outside [that hollow], it can perceive more than half of the mirror, that portion being defined by the circle on the sphere upon which two rays drawn from the center of sight are tangent.¹¹⁷

[5.47] Now, if the eye lies at the center of this mirror, reflection will not take place from any point other than [that according to which the form of the eye reflects back] to itself, for any line extended from the center of the sphere to the [surface of the] sphere is perpendicular to the plane that is tangent to the sphere at that point. Hence, in this situation the eye will see nothing but itself through reflection.

[5.48] If, however, the eye is located outside the center of the sphere, reflection can occur to another body from any point on the mirror other than the point where the diameter drawn from the center of sight to the sphere passes through the sphere's center, because the diameter falls perpendicularly to a plane that is tangent to the sphere. Now, having chosen another point [on the surface of that sphere], draw a diameter [normal] to it from the center of the sphere, and [draw] a line from the center of sight [to it]. Accordingly, an acute angle will be formed by these lines, for the line of sight lies between the diameter [that forms the normal] and the plane tangent to that point, and that plane lies [on the] outside [surface of] the sphere. And whether the center of sight lies inside or outside the sphere, this line of sight falls inside the mirror because it falls between the lines of sight that are tangent to the circle [defining] the [visible] portion of the sphere when the center of sight lies outside [the sphere].

[5.49] If the center of sight lies inside the sphere, it is obvious that the line of sight falls inside the mirror. Accordingly, since the diameter [that forms the normal] forms a right angle with the tangent [to the chosen point], cut off from that [right angle] an acute angle equal to the aforementioned

acute angle in the same plane. I say, then, that the line of reflection falls inside the mirror, because the common section of the mirror's surface and the plane of reflection is a circle that forms with the diameter [constituting the normal] an acute angle greater than any acute rectilinear angle, and in each such point there will be reflection of this kind.

[5.50] It is evident from these observations that in every plane of reflection there will be a center of sight, the center of the mirror, a point of reflection, a point that is seen, and the endpoint of the diameter extended from the center of sight to the [surface of the] sphere through the sphere's center. Furthermore, the common section of all these planes with the surface of the mirror is a circle, and reflection can occur in the same plane from any point on this common section.¹¹⁸

[5.51] In the case of concave cylindrical mirrors the entire mirror can be perceived if the center of sight lies inside it. However, if it lies outside, more than half the mirror will be visible, the portion, that is, that lies between the two planes projected from the center of sight and tangent to [the surface of] the cylinder.¹¹⁹

[5.52] We will imagine a plane projected from the center of sight and parallel to the bases of the cylinder. This plane will either intersect the cylinder or it will not. If it intersects, then the common section of this plane and the cylinder will be a circle, and the line of sight passing through the center of this circle will fall orthogonally to the plane tangent to the cylinder at the point where the line of sight intersects it. So reflection will take place along the same line to its source.

[5.53] Let some other point be chosen. The line extended normal [to the cylinder's surface] from this point will intersect the axis, and the line of sight dropped to this point will form an acute angle with the [aforementioned] normal, since it lies between that normal and the [line] tangent [to the cylinder's surface at that point]. And that this line will fall inside the mirror is obvious from the fact that it falls between the planes that are tangent to the visible portion [of the cylinder]. Hence, in the same plane of reflection we can cut off from the angle formed by the normal and the tangent an acute angle equal to the aforementioned acute angle. Moreover, the line of reflection forming this angle will fall inside the cylinder, because it will fall between the normal and the line along the length [of the cylinder] passing orthogonally through the endpoint [of the flanking, tangent line of sight]. Thus, the center of sight, the point of reflection, the point that is seen, and the point on the axis to which the normal falls will [all] lie in the plane of reflection.¹²⁰

[5.54] Moreover, if the center of sight is located in such a way that the common section of the plane of reflection and the surface of the cylinder constitutes a line along [the cylinder's] length, reflection may occur from

any point on that common section. The plane common to all these reflections will be in one specific plane—i.e., the one in which the center of sight and the entire axis of the cylinder lie—as was said earlier in the case of the non-concave cylindrical mirror.

[5.55] Likewise, if the common section is a circle, all of the reflections occurring from points on that circle will proceed in the same plane, as was shown earlier for the other circles.

[5.56] But if the common section is a cylindric section [i.e., an ellipse], reflection will occur from only two of its points within the same plane, although in the case of [convex] cylinders discussed above, [when the points of reflection lie at the intersection of the ellipse and a] circle, reflection would occur from only one point in a single plane when one eye is looking, because in the previous case the corresponding points of intersection through which the circle parallel to the bases passes were invisible to the eye. For when one [such point] was seen [with one eye], the other was invisible because a portion smaller [than half] of the cylinder is visible, but in these cases [when the cylinder is concave], a greater portion [than half] of the cylinder is visible, so the [corresponding] points [of intersection] of the circle parallel to the bases and the common section are perceived by one eye.¹²¹

[5.57] In the case of concave conical mirrors, if the center of sight lies inside the mirror, it will see all of it. If, however, it lies outside, and if the line extended from a center of sight [located above the mirror] to the vertex of the cone enters the cone or coincides with a line along the length of the cone, nothing of the mirror will be seen. For any other line extended from the eye to the cone will fall on the cone's outer surface, so the inner surface will be hidden.¹²²

[5.58] If, however, a segment is removed from the cone, the portion of the cone lying between the planes projected from the center of sight and tangent to [the cone's] surface—i.e., more than half—can be seen, and if the line from the center of sight is perpendicular to the plane tangent to the cone and reaches to the axis, then, as was said in the case of the other cones [i.e., convex cones], the common sections [of the plane of reflection and the mirror's surface] will be either [straight] lines along the length of the cone or [conic] sections.¹²³ Moreover, in these cases [i.e., when the plane of reflection is a conic section and more than half the surface of the mirror is visible], reflection can occur from two points on the [conic] section in the same plane with respect to the same center of sight; and the center of sight, the point seen, the point of reflection, and the point on the axis will lie in the same plane of reflection.

[5.59] However, if an entire conical mirror faces the center of sight, and the center of sight lies on the side of the base, only what lies inside the mirror will be perceived, because the normal forms an acute angle with the

line drawn to it from the center of sight on the side of the base. Thus, reflection occurs on the side of the vertex, and all of the reflected lines will fall within the cone, and [only] what is placed within the cone can be seen.¹²⁴

[5.60] On the other hand, if a segment of the cone is cut off lengthwise, things outside the mirror can be perceived, since what lies beyond may be open to the lines of reflection. Likewise, if the cone is truncated in the form of a ring with the vertex removed, lines [of incidence] will have free entry [through that upper opening], and things outside the mirror will be visible. And if the center of sight lies inside the concave [inner surface of the mirror], more things that lie outside can be seen than [if the the center of sight lies] on the side of the base, because it provides a wider scope for the lines of reflection to project outward.¹²⁵

[5.61] Furthermore, given a point on any mirror, it is possible for the form of only one point to be perceived at it by the same center of sight. For only one plane passes through the normal and the center of sight, and there exists only one [straight] line between the center of sight and the [given] point, and there exists only one acute angle formed by that line and the normal, and in the same plane there is only one acute angle equal to this one, so there is only one line that forms an angle with the normal that is equal to this one. So when the line [of incidence] is extended to a point on a [visible] object, the form of another point [on that object] cannot be conveyed along it, since the [form of the] first point passing along it will block any following it. But two point-forms can be perceived at the same point on the mirror by both eyes, because an infinite number of cutting planes can be imagined along the normal, and in each of them two equal acute angles can be imagined with respect to the normal.

[5.62] We have therefore now explained the character of reflection as well as the character of each [type of] mirror. When sight perceives forms by means of reflection, it is unaware that this perception is due to reflection. For reflection is not a function of sight, because, when the eye is removed, the form nonetheless radiates from the [visible] object to the mirror, and it will be reflected in the way just described. But if the eye happens to lie where the lines of reflection come together, then sight will perceive that form at the endpoints of these lines, and that form [appears to] exist in the mirror, not as if it were adventitious but as if it were a natural form [actually] in the mirror. Moreover, sight sometimes perceives forms in mirrors on the surface alone, sometimes in front of the mirror, sometimes beyond it. And the place where the form appears will depend on the shape of the mirror and the location of the visible object, and the form is always perceived in its appropriate place when the [relative] situation of the center of sight and the mirror changes. And the distance of the form's location from

the mirror will vary with variation in the shape of the mirror. And the location of the form is called image-location, and the form is called an image.¹²⁶ Moreover, the eye perceives the visible object at the image-location, and we shall discuss this location and its essential characteristic for each [type of] mirror, and we will enumerate and explain the reasons visible objects are perceived at that location, and we will do so in the following book, God willing.

NOTES TO BOOK FOUR

¹The characteristics to which Alhacen is referring here are the twenty-two visible intentions that he deals with in detail in book 2 of the *De aspectibus*; see Smith, *Alhacen's Theory*, 438-512.

²This tripartite categorization of vision is clearly articulated in Ptolemy's *Optics*, although its wellsprings lie much earlier in Greek optics, particularly with the study of refraction by Archimedes. Note the emphasis on *vision* rather than on the action of the rays (whether visual or light) in this threefold breakdown of the science of optics.

³That is, aside from all the visual deceptions pertaining to direct vision discussed in book 3, sight is subject to particular deceptions in reflection and refraction. In both cases, for instance, the image and the object will lie at different locations (e.g., behind the mirror), and the object may appear larger or smaller than it should (e.g., magnified or diminished). In reflection, moreover, the object may appear distorted in shape and orientation depending on the type of mirror. As far as reflection is concerned, these distortions are dealt with in book 6.

⁴The fact that light and color radiate from any luminous or illuminated body to any facing body is discussed in detail by Alhacen in chapter 3 of the first book of the original Arabic version of the *De aspectibus*, where he establishes the punctiform and rectilinear radiation of light and color; see Sabra, *Optics*, vol. 1, 13-51. Along with the first two chapters of the Arabic original, this chapter is missing from the Latin version; see Sabra, *Optics*, vol. 2, lxxvi-lxxvii, and Smith, *Alhacen's Theory*, ix. However, the basic point that Alhacen makes in this chapter—that light and color radiate along straight lines in all possible directions from any point on a luminous or illuminated surface—is clearly implicit in his subsequent analysis of light and vision in books 1-3 of the Latin version of *De aspectibus*.

⁵The luminosity inherent to an illuminating object is primary, whereas the light in a body illuminated by that object is secondary. Thus, the light of the sun is primary with respect to the light of the moon, but the light of the moon is primary with respect to the light on objects illuminated by moonshine. As is clear from these examples, secondary light is fainter than primary light. For Alhacen's discussion of these two types of light, see Sabra, *Optics*, vol. 1, 13-40.

⁶Throughout book 4, Alhacen describes mirrors made of only two materials, iron and silver. Most of his experiments specify iron mirrors, although he suggests silver ones when greater reflectivity is needed. Later on in book 4, Alhacen describes the formation of convex and concave spherical, cylindrical, and conical iron mirrors with very short focal lengths. Given the extreme difficulty of molding and working iron, Alhacen's reliance on such mirrors indicates an extraordinarily high—improbably high, in fact—level of technical proficiency among iron workers in early eleventh-century Cairo, where Alhacen presumably carried out the experiments described in *De aspectibus*; see pp. xxv-xxiv above for some discussion of this point.

⁷Daylight, according to Alhacen, is the secondary light in the atmosphere that results from its being illuminated by the sun. See Sabra, *Optics*, vol. 1, 23-29.

⁸In this experiment, as well as in subsequent ones carried out in the same room, Alhacen's description of the requisite controls is less precise than we might wish. Granted, the room has one window, but how large should it be? How far from the window should the mirror be placed? How far from the mirror should the directly illuminated object be placed? The size of the window is perhaps the most pressing issue, and it is reasonable to suppose that Alhacen intended it to be quite small so that the light shining through it would be narrowly channeled into the room, leaving most of it in shadow. Accordingly, figure 4.2.1, p. 191, offers a schematic reconstruction of the experiment. C and D represent two objects placed in a room with one window through which daylight shines. Since Alhacen does not specify the shape of these objects, I have represented them as spheres to stand generically for whatever objects he may have in mind. C, which is presumably white (although Alhacen makes no such specification), is directly exposed to the incoming daylight. D is blocked from such exposure, either because it lies outside the area of effective illumination or because it is shielded by the mirror at F, which is represented as plane, although again Alhacen makes no specification. This mirror is placed in such a way that the secondary light radiating from C is reflected to D, thus faintly illuminating the part of its surface exposed to the reflected light. If D is moved from its original location while kept from direct exposure to the daylight shining through the window, it will lose the illumination previously imparted to it, although it will do so progressively rather than suddenly, since the light from C is reflected from the entire mirror.

⁹Under the same conditions specified in the previous experiment, a third white object, E in figure 4.2.2, p. 191, is placed on the other side of mirror F out of the way of direct illumination through the window. To ensure that E receives no illumination through the window, one might put a screen in front of the object, although Alhacen makes no mention of this expedient. The presumed point of this experiment is to show by contrast with the relative darkness of E that D is indeed illuminated by C's reflected light. A silver mirror is suggested in this case to heighten the effect of the reflected illumination on D.

¹⁰For this demonstration, see I, 3.99-103, in the Arabic original of the *De aspectibus*, which is missing in the Latin version; see Sabra, *Optics*, vol. 1, 40-41.

¹¹There is no distinction between "accidental" and "secondary" light insofar as both are extrinsic rather than intrinsic sources of illumination in the given body; see, e.g., I, 3.88, in Sabra, *Optics*, vol. 1, 38.

¹²The experiment suggested here is the same as that illustrated in figure 4.2.1, p. 191, except that object C is brightly colored rather than white, and it is exposed to direct sunlight rather than daylight. The goblet Alhacen has in mind here is no doubt the clear glass one he describes earlier in I, 3.105 (Sabra, *Optics*, 1, 41) and 1, 4.22 (Smith, *Alhacen's Theory*, 347). Why object D, to which the luminous color on the surface of object C is reflected, is placed inside such a goblet is left unexplained, but Alhacen's intent in placing the object there may be to have it suspended in air without the interference of a hand or some other opaque body to hold it in place.

¹³Alhacen's point here follows from his discussion of color-radiation in 1.6.95-99, in Smith, *Alhacen's Theory*, 381-382. When a colored body is exposed to light so that the color becomes illuminated, the resulting luminous color radiates. But the illumination in that body is already secondary, so its resulting illumination, along with the color, is essentially tertiary. Accordingly, the radiated form of the color is commensurately weaker than the illuminated color serving as its source, and if the color is already weak (i.e. pale or muddy), its radiated form will be all the weaker when it reaches the mirror.

¹⁴As illustrated in figure 4.2.3, p. 192, the setup for this experiment is essentially the same as that for the earlier ones carried out in the room with one narrow window. White object C is exposed to light radiating through the window, and white object D is placed toward mirror F to catch the reflected light from C. White object E is placed near the mirror in such a way that it is exposed to the secondary light radiating directly from C. The illumination on the face of E will be brighter than that on the face of D. To ensure that no direct light from the window reaches E, one might place a screen in front of E. Likewise, to ensure that secondary light from C does not reach D, one might place a screen between the two bodies. This second screen would seem to be crucial to the success of the experiment because, if secondary illumination were allowed to reach D directly from C, that illumination would be added to the illumination reflected to D. Alhacen, however, makes no mention of either expedient, which is somewhat surprising given the punctilio with which he describes the experimental conditions for his empirical analysis of reflection later in the third chapter.

¹⁵In this case, object C in figure 4.2.3, p. 192, is brightly colored, rather than white. Accordingly, the luminous color from object C will radiate directly to object E while reflecting to object D, the coloring effect on object E being brighter than that on object D. The vessel mentioned here is presumably the (glass) goblet mentioned earlier. Alhacen does not mean to deny that the color of the iron mirror has a darkening effect on the light or luminous color reflected from it; in fact, he discusses this effect in the next paragraph. What he is denying is that this effect is the sole, or even primary, cause of the weakening of light and color in reflection.

¹⁶Clearly, Alhacen understands the distance between the window, as source of illumination, and the elevated body to which light is reflected as a composite of the lengths of the incident and reflected rays. Hence, that distance, overall, should be equal to the uninterrupted distance between the window and the body lying on the ground.

¹⁷That is, the clear glass vessel mentioned earlier.

¹⁸Here Alhacen makes explicit that the distance between the source (i.e., the brightly colored object illuminated by sunlight) and the body to which light is reflected is composed of the lengths of the incident and reflected rays.

¹⁹In specifying that the object illuminated by secondary light should lie the same distance from the mirror as the object to which the light shining directly through the window onto the mirror is reflected, Alhacen seems to have in mind that the object be placed behind the mirror at the appropriate distance so as to be blocked from direct illumination. In addition, although Alhacen does not stipulate

whether the light coming through the window should be direct sunlight or daylight, the experiment as described would seem to be more suited to direct sunlight. Accordingly, in the first case, with the mirror large enough to catch all the light radiating through the window, the object behind the mirror would be blocked not only from direct sunlight but also from secondary illumination cast on the ground around the mirror and radiated to it. In that case, of course, the contrast in brightness between the object illuminated by reflection and the body shielded from both direct and secondary illumination would be obvious. In the second case, with the mirror small enough to allow some of the incoming light to illuminate the floor behind it, or to illuminate something white, such as a sheet, on the floor behind it, the object behind the mirror will still be blocked from direct sunlight but will be exposed to secondary radiation from the floor or the sheet. Nonetheless, the illumination on the object receiving the reflected sunlight will be more intense than that on the body receiving the secondary illumination.

²⁰In tying reflectivity to the physical structure of the reflecting surface, Alhacen is laying the ground for his subsequent dynamic explanation of reflection by analogy to the physical rebound of a swiftly moving projectile (see paragraphs 3.99-3.102 below). That Alhacen does not conceive of light-radiation as true projectile motion is clear from his insistence that what passes from light sources is formal rather than material, yet common experience shows that certain bodies owe their polish to physical burnishing, which smooths their surfaces. Hence, light seems to act according to the kinetic and dynamic principles that govern the motion of physical bodies.

²¹Here Alhacen is laying out the basic agenda for book 4: to show both empirically and mathematically that reflection from any reflecting surface, no matter its shape, occurs in a plane that is orthogonal to that surface, that the way reflection occurs is governed by the relationship between the normal to the point of reflection and the lines of incidence and reflection, and that this relationship is governed by the equal-angles law.

²²Throughout the remainder of book 4 and beyond, Alhacen describes convex mirrors as "polished on the outside" (*extra/exterius politum*) rather than as *convexum*. Concave mirrors, on the other hand, he sometimes describes as *concauum* and sometimes as "polished on the inside" (*intus/interius politum*).

²³As explained earlier in book 3, a "digit" is approximately $3/4$ " or 1.9 cm; in fact, it is somewhat less, so the plaque is roughly 4.5" by 9", or 11.3 cm. by 22.5 cm. This we know because of his specification of 4 digits for the width of a board used for testing diplopia, that distance, according to Alhacen, being the distance between the pupils of the two eyes; see Smith, *Alhacen's Theory*, 573. Alhacen's choice of measurement here is arbitrary; the crucial thing is that the plaque be twice as long as it is wide. Nevertheless, once the measurements are fixed at 12 by 6, the rest of the construction is contingent on that measure.

²⁴Figure 4.3.1, p. 192, which illustrates the final scribing on the plaque, is found only in mss *P1*, *S*, *E*, *O* and *L3* and therefore appears not to be a canonical part of the original text. Among those manuscripts, moreover, the diagram appears in various forms, none of which agrees with that of Risner in his edition of *De aspectibus*.

In order to hew as closely to the manuscript-tradition as possible, I have followed the basic form and lettering of the versions found in mss *E* and *L3*, which are closely related. According to Alhacen's description, side *DB* of the bronze rectangle is 12 digits long, while side *AB* is 6. At midpoint *E* of edge *DB* a semicircle is drawn with a radius of 6 digits to pass through points *B*, *F* and *D*. Radius *FE* is drawn perpendicular to *DB*. Point *G* is marked on *FE* at a distance of 1 digit below *F*. Again from point *E*, a semicircle of radius *GE*, i.e., 5 digits, is drawn. Then, points are chosen on arcs *FB* and *FD* so that corresponding lines passing through them, such as *HE* and *ME*, or *EA* and its mate on the other side, will form equal arcs and thus equal angles at *E*. Finally, line *OV* is drawn 1 digit above and parallel to *DB* so as to cut lines *EH* and *EA* at points *L* and *K*, respectively.

²⁵Figure 4.3.2, p. 193, represents the plaque with all the excess cut away. The instructions for cutting away the lower portion are open to interpretation because it is unclear whether the radius to which the lines are closest is *FE* or *DE* and its correspondent *EB* on each side of *E* in figure 4.3.1, p. 192. As I have represented it here—and Risner as well—the cut is made along the last of the three lines, i.e., the one beyond *EA* and its mate on the other side.

²⁶Mss *O* and *E*, as well as Risner, have diagrams representing the instrument that is to be formed according to the initial description here. However, in order to make its overall construction easier to follow, I have illustrated the successive stages in that construction with my own diagrams. Illustrated in figure 4.3.3, p. 193, is the initial scribing of the top of the block, which is 14 digits on each side. The outer circle has a radius of 7 digits, the inner one a radius of 5 digits. Line *AE* on the top of the block corresponds to line *EF* on the bronze plaque in figure 4.3.2, p. 193, line *BE* on the top of the block to line *HE* on the plaque, and so forth. Hence, angle *AEB* on the top of the block is equal to angle *FEH* on the plaque, and so forth.

²⁷Figure 4.3.4, p. 194, represents a bird's-eye view of the ring with the remnants of the originally scribed lines on the top surface, which is 2 digits wide, the outer diameter of the ring being 14 digits and its inner diameter being 10 digits.

²⁸As specific as it sounds, this procedure is so vague as to be a matter of sheer conjecture. The main problem lies in the meaning of *acuta* as applied to the ruler Alhacen describes. On the one hand, he may have in mind the ruler represented on the left-hand diagram in figure 4.3.5, p. 194, where the lower edge is marked off in equal sections and the ruler actually comes to a point. On the other, he may have in mind the ruler represented in the right-hand diagram, in which *acuta* would best be translated as "sharp-edged." In my reconstruction of the procedure described in this passage, the measuring edge of the ruler is applied to the outer endpoint of one of the lines at the top of the ring and held firmly in place while the ruler is slid along the outer surface until the point touches that surface. Accordingly, the line between the spot where that point touches the ring's outer surface and the endpoint of the line at the top of the ring will be perpendicular to the top edge of the cylinder. Suffice it to say, there is considerable leeway for error in such a procedure; yet, as will become clear later on, the final construction of the instrument based on the scribings described to this point requires extraordinary exactitude.

²⁹The procedure here is clear enough in theory, but in practice Alhacen offers no method for finding the midpoint of the semicircle's arc with precision, although

this could be done easily, albeit indirectly, by recourse to Euclid's *Elements*, III.3. It is puzzling that Alhacen did not fall back on the simple schoolboy expedient of drawing intersecting arcs from equidistant points on either side of the point to which the perpendicular is to be dropped, a method based on *Elements*, I.11.

³⁰As a standard measure, a grain of barley = approximately $1/3$ in or .85 cm., so half a grain of barley = approximately $1/6$ in or .42 cm.

³¹Figure 4.3.6, p. 195 shows a cutaway view of the ring with the vertical lines along the inside surface cut by the cavity, whose upper edge lies half a grain of barley below the 2-digit level from the bottom, which is represented by the faint line passing through points A, B, and C of the midmost and the two flanking lines of longitude on the inner surface of the ring.

³²Although Alhacen does not explain precisely how the bronze plaque is to be inserted, it obviously cannot be done unless the ring is cut in such a way as to allow the plaque to be fit into the cavity. Presumably the two sections would then be glued back together.

³³This procedure ensures that the axis of each cylindrical hole will lie exactly half a grain of barley above the top surface of the inserted bronze plaque. Figure 4.3.7, p. 195, gives a cutaway view of the ring with the plaque inserted and the holes drilled through the cylinder's wall.

³⁴The scribing of lines described to this point is illustrated in figure 4.3.8, p. 196, with the central square formed by the intersection of the outer parallel lines being 4 digits on a side. The space between the outer and inner circles is the same size as the space between the outer and inner edges of the ring. This figure occurs in various forms in mss *P1*, *O*, *L3*, and *E*.

³⁵The block, with the square cavity, is represented in figure 4.3.9, p. 196.

³⁶Figure 4.3.10, p. 197, represents a bird's-eye view of the ring, with the bronze plaque inserted and the block at the bottom attached. Note that the point of triangular section of the bronze plaque lies directly over the midpoint of the square cavity in the bottom block.

³⁷Presumably, the parallel line to which Alhacen refers is line OV in figure 4.3.2, p. 193. Evidently it does not matter which of the lines converging toward E is chosen for the determination of the tube's length as long as that length remains fixed.

³⁸The construction of all seven mirrors is illustrated in figure 4.3.11, p. 198. The plane mirror forms a circle whose diameter is 3 digits and whose thickness is indeterminate, but probably fairly small. The concave and convex cylindrical mirrors are formed from a hollow cylinder that is 3 digits high and 6 digits in diameter at the base. The cylinder is cut along chord AB, which is 3 digits long, leaving axis DF of the excised segment less than half a digit long. This follows from the fact that, being half the diameter of the circle, chord AB subtends an arc of 60° . Hence, $CD = 3(\cosine 30^\circ) = 3(.866) = 2.6$, so $DF = 3 - 2.6 = .4$. Likewise, the spherical concave and convex mirrors are formed from a hollow sphere whose diameter is 6 digits. This sphere is cut along chord AB so that the resulting mirrors will be segments of that sphere having a circular base with a diameter of 3 digits and an axis roughly .4 digits long. The conical mirrors, finally, are formed from a hollow right cone whose

base is a circle with a diameter of 6 digits and whose line of longitude FV is 4.5 digits long, which means that the cone is just over 3.3 digits high along its axis from vertex to base. The resulting mirrors consist of the segment ABV cut along lines AV and BV.

³⁹Although Alhacen does not specify the thickness of the panels, it is clear from this instruction that they can be no more than 2 digits thick if they are to fit upright in the square cavity in the block at the base of the ring while touching the point of the triangular section of the bronze plaque inserted into the ring.

⁴⁰Figure 4.3.12, p. 199, which derives from diagrams in mss *P1* and *E*, represents one of the panels described here. Being 4 digits wide and no more than 2 digits thick, it is designed to fit snugly into the square cavity in the block at the base of the ring yet stand upright facing the point of the bronze plaque. That point touches the inserted panel at point X along the midline of longitude AB, which is 6 digits long. Hence, the distance from the base of the panel to point X consists of the 1 digit to which the panel is sunk in the square cavity plus 2 digits minus the half a grain of barley that the bronze plaque lies below the 2-digit mark above the bottom surface of the ring. If a distance of half a grain of barley is marked above point X, the resulting point D will therefore lie precisely 3 digits above point B and will be the midpoint of line of longitude AB. Bisecting the two segments of that line lying above and below point D will therefore yield four sections of 1.5 digits along it.

⁴¹The insertion of the plane mirror is represented in the top diagram of figure 4.3.13, p. 199, which provides a face-on view of the panel on the left, a side view of the panel on the right, and a view from the bottom of the panel below.

⁴²The insertion of the convex cylindrical mirror is represented in the lower left-hand diagram of figure 4.3.13, p. 199, according to the same three views as before.

⁴³The insertion of the concave cylindrical mirror is represented in the lower right-hand diagram of figure 4.3.13, p. 199, according to the same three views as before.

⁴⁴The insertion of the convex conical mirror is represented in the upper left-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the mirror's line of longitude, VF (as represented in figure 4.3.11, p. 198), coincides with the midline along the length of the panel and is therefore in the same plane with it. Notice, too, that since the mirror is formed from a right cone, and since its line of longitude makes a sharply acute angle with the base, the arc on the base of the section inserted into the panel will be visible as represented, assuming that the sides of the cavity into which the mirror is inserted are perpendicular to the plane of the panel's face.

⁴⁵The insertion of the concave conical mirror is represented in the upper right-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the mirror's line of longitude VF does not coincide with the midline along the length of the panel but is parallel to and below it. Note also that the arc at the base will be visible as represented, assuming that the sides of the cavity into which the mirror is inserted are perpendicular to the plane of the panel's face.

⁴⁶In this case the ruler prescribed by Alhacen would seem to be of the kind represented on the right of figure 4.3.5, p. 194. The insertion of the convex spheri-

cal mirror is represented in the lower left-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the midpoint of the mirror's surface coincides with the midpoint of the panel's surface and thus lies in the same plane with it.

⁴⁷The insertion of the concave spherical mirror is represented in the lower right-hand diagram of figure 4.3.14, p. 200, according to the same three views as before. Note that the midpoint of the mirror's surface lies below but directly in line with the midpoint of the panel's surface along the mirror's axis.

⁴⁸Although Alhacen's description of the ruler and its use is far from clear in detail, it seems clear enough in a general way. The first step in the procedure is done with the panel outside the ring. Accordingly, as represented in the left-hand diagram of figure 4.3.15, p. 201, the sharply pointed ruler is poised with its measuring edge along midline AB of the panel so that its point lies directly upon the panel's midpoint D and, therefore, directly above the midpoint of the mirror's concave surface. A needle is dropped orthogonally from the ruler's point to the mirror's surface so as to touch its midpoint, and the spot where the needle intersects the point of the ruler is marked. The distance from this mark to the needle's point is thus the length of the mirror's axis—i.e., approximately .4 digits. This distance is then marked off on radius GE of the bronze plaque from point E of the triangle on the plaque to form XE, as represented in the right-hand diagram of figure 4.3.15, which is drawn out of scale in order to make things clearer.

⁴⁹What was set out to be shown, of course, is that the point of the bronze plaque does not reach all the way to the plane of the mirror's centerpoint since it strikes the mirror in front of that plane. As represented in the right-hand diagram of figure 4.3.15, p. 201, the mirror is inserted upright to face the point of the bronze plaque and is then brought up to the plaque until its point touches the mirror at E. The ruler is dropped along midline AB of the panel until it reaches radius GE, which passes through the vertex of the triangle on the bronze plaque. Point Y, where the ruler's point touches GE, is marked. Since the bronze plaque lies half a grain of barley below the mirror's axis DF (as represented by distance DY, which is grossly exaggerated for purposes of clarity), distance YE will be shorter than distance DF; or, as Alhacen puts it, E will be higher than F, the actual midpoint of the mirror.

⁵⁰Having marked point E on the concave mirror, as represented in the right-hand diagram of figure 4.3.15, p. 201, we then bore a hole there so that the vertex of the bronze plaque's triangle can be pushed through the mirror until point X on radius GE of the plaque reaches the midline along the length of the panel—distance XE, which is the height of the mirror's axis, having already been marked on radius GE of the bronze plaque. In thus moving distance XY, the point of the plaque's triangle will reach point E' behind the mirror, EE' thus being equal to XY. Accordingly, line FE' will lie in a plane parallel to the surface of the panel, so it will be perpendicular to line GE and thus to the surface of the plaque. The point of all this will become clear shortly, when Alhacen sets up the apparatus with the panels inserted to demonstrate empirically that light reflects at equal angles from any mirror, no matter its shape.

Alhacen's full description of the construction of the bronze plaque, the wooden ring, the panels, the mirrors, and the insertion of the mirrors is extraordinarily detailed in some ways and extraordinarily vague in others. For instance, although he describes the insertion of the mirrors in exact detail as far as measurements are concerned, he offers no description of precisely how to cut out the cavities in the panels that will accommodate the mirrors. Furthermore, the constructions described by Alhacen require an astonishing exactitude that offers very little, if any, tolerance for error. Particularly salient in this regard is the description of how to accommodate the point of the bronze plaque to the concave spherical mirror. We know that the radius of the mirror is 3 digits, which translates to roughly 2.25 in or 6 cm. We also know that distance FE' in figure 4.3.15 is half a grain of barley, which translates to roughly .17 in or .42 cm. Therefore, chord EH will be approximately .33 in or .84 cm. and, according to the measure of the radius, will subtend an arc of approximately 13° . On this basis, we arrive at a figure for EE' of around .0064 the mirror's diameter, which comes to roughly .014 in or .036 cm., which = .36 mm. Not only is it difficult to believe that Alhacen could have achieved such exactitude given the available technology, but it is doubtful that an adjustment of such negligible magnitude would have made any real difference in the accuracy of the overall experimental setup. This leaves open two possibilities: either the technological resources available to Alhacen were far more sophisticated than expected, or the entire setup, including the ring, the mirrors, and the panels, was only imagined by Alhacen and never actually constructed.

⁵¹Why Alhacen covers all the holes but the one through which the light passes with parchment (presumably bright white parchment) is clear at this point: i.e., to make the illumination created by the reflected light as clear and bright as possible. The reason for marking the outline of the holes on the parchment will become clear later, when Alhacen shows that the reflected light disperses in the form of a cone rather than proceeding along a perfectly straight shaft with the same diameter as the hole through which it enters the ring. Two things are worth noting at this point. First, the reason the plaque is placed so low in the cylinder is to ensure that the area around it is relatively unaffected by ambient light and thus relatively dark. This, of course, highlights the effect of the reflected light. Second, if enough particles of smoke, dust, or the like are suspended in the air inside the hollow of the ring, the beams of incident and reflected light will actually be visible.

⁵²The point of these procedures with the tube is, of course, to narrow the beam of light as much as possible so as to reduce it to something approaching a mathematical ray. In addition, since the iron tube reaches quite close to the mirror (see paragraph 3.22 above), the beam projected by it onto the mirror will have little chance to spread out, as light is naturally apt to do.

⁵³Alhacen's point in comparing the three circles—i.e., the circle at the base of the hole, the circle projected onto the mirror, and the circle projected by reflection to the inner surface of the cylinder—is to show that, no matter how much we narrow the beam, light propagates conically, so the base of the cone continually expands the farther the light propagates. Given the small size of the hole through which the light passes, and given how close the mirror is to that hole, determining

the difference in size between the circle of illumination on the mirror's surface and the circle at the inner end of the hole would be extremely difficult. Like the construction of the setup, this procedure demands an astonishing exactitude.

⁵⁴The point here is to show that the expansion of light described above is not due to the axial ray's reflecting at different points on the circle of illumination projected onto the mirror. In other words, the ray imagined to follow the axis of the hole will reflect from only one point on the mirror, the point where the axis of the hole intersects the mirror.

⁵⁵Although the direction of slant is not clear from this description, it seems clear subsequently that the panel is to be slanted backwards so that the reflected light is projected upward, toward the top of the ring, rather than downward, toward its base, because in this latter case the light would be projected onto the bronze plaque rather than to the inside wall of the cylinder. One clear implication of this is that the incident and reflected rays lie in a plane that is perpendicular to the surface of the mirror as well as to the surface of the bronze plaque.

⁵⁶The point of this procedure is to show that, no matter how the mirror is slanted to the left or right, as long as the ray striking it is normal to the plane tangent to its line of longitude—i.e., the plane of the panel's face—the light will reflect back along that normal. Moreover, as Alhacen goes on to show, this fact applies to the concave cylindrical mirror as well as to the convex and concave conical mirrors. Given that the panel is 4 digits wide and that the cavity is also 4 digits wide, and assuming that both measurements are as exact as possible, slanting the panel to the right or left for the sake of this experiment would seem to be impossible, although indeed the results of the experiment are intuitively obvious. Here we have yet another indication that the entire framework of these experiments was imagined rather than actually executed.

⁵⁷At this point the rationale for boring the hole in the concave spherical mirror to accommodate the point of the bronze plaque becomes clear: namely, to ensure that the distance from the centers of the holes to the midpoint of the mirror—i.e., the point of reflection—is precisely the same as the distance of the point on the bronze plaque to the bottoms of the holes. That in turn ensures that the axial rays reaching the midpoint of the mirror will be perfectly parallel to the lines scribed on the surface of the bronze plaque and intersecting at its point.

⁵⁸In other words, if FE' in figure 4.3.15, p. 201, represents the midline along the longitude of the panel containing the convex spherical mirror whose midpoint is F , then the ruler is placed with its point on F and the panel is moved until the point of the bronze plaque, which is now taken to be at E' along perpendicular radius $G'E'$, lies directly in line with the point of the ruler. Thus, when the ruler is lowered until both points touch, the panel will be positioned so that the midpoint of the mirror and the point on the bronze plaque lie on line FE , just as was the case for the concave spherical mirror, and for precisely the same reason.

⁵⁹Thus, Alhacen is extending to reflection what he already established in his analysis of direct radiation: that light invariably acts in the same way, whether it is primary or secondary.

⁶⁰The points made in this paragraph are illustrated in figure 4.3.16, p. 201, where the ellipse represents the common section formed by the plane of the top surface of

the bronze plaque, VHGM O, and the inside surface of the ring. EE'' represents the midline of longitude along the inserted panel. GE represents the bisecting radius of the plaque, G''E'' the midline of the bottom surface of the square cavity dug into block at the base of the ring, and G'E' the midline of the top surface of the block at the base of the ring. X''Y'' represents the common section of the bottom surface of the cavity in the block and the face of the inserted panel, X'Y' the common section of the top surface of the block and the face of the inserted panel, and XY the common section of the top surface of the plaque and the surface of the panel inserted into the square cavity in the block. By construction, the plane of the floor of the cavity in the base-block, the plane of the top surface of the base-block, and the plane of the top surface of the bronze plaque are parallel, so the midlines G''E'', G'E', and GE within those respective planes will be parallel. Moreover, since the panel with its mirror is inserted upright within the cavity in the base-block, all three of those planes will intersect that panel orthogonally along common sections X''Y'', X'Y', and XY from the bottom up. So the midlines within those planes will be perpendicular to those common sections as well as to midline EE'' of the panel.

⁶¹In figure 4.3.17, p. 202, VHGM O represents the plane of the top surface of the bronze plaque, GE being the radius of bisection, and HE and ME being corresponding flanking lines on the plaque's top surface. Points A, C, and B represent the centerpoints of the holes at the inner surface of the ring (drawn out of scale for clarity's sake), and AR, CR, and BR represent the axial rays passing through those holes to intersect at R, which lies on midline RE' along the length of the panel. Accordingly, the plane formed by those axial rays is parallel to the top surface of the bronze plaque, so those axial rays are all perpendicular to RE', which is perpendicular to the top surface of the bronze plaque. Likewise, lines AM, CG, and BH form segments of the lines of longitude scribed on the inner surface of the ring during its construction, so they are perpendicular to the top surface of the bronze plaque as well as to the plane of the axial rays, so they are all parallel to RE. Moreover, AM, CG, and BH are radii of their respective holes, A, C, and B being the centerpoints, and M, G, and H being the points where those holes are tangent to the top surface of the bronze plaque. Accordingly, RE, AM, CG, and BH are all equal (i.e., half a grain of barley), and, since they lie between equal and parallel lines, so are AR and ME, CR and GE, and BR and HE, all of which are equal to one another, because the lines scribed on the bronze plaque are all radii of semicircle VHGM O.

⁶²In other words, since angles GEM and GEH on the bronze plaque represented in figure 4.3.17, p. 202, are equal by construction, then angles CRA and CRB must be equal according to the parallelism and equality of the corresponding lines in the axial plane.

⁶³Here, then, Alhacen has established that the plane of reflection, which contains the normal and the incident and reflected rays is orthogonal to the reflecting surface or the plane tangent to that surface at the point of reflection.

⁶⁴Alhacen's point here is illustrated in figure 4.3.18, p. 202. From any random point A, G, or B on the luminous surface represented by AB, a cone of radiation is formed with its base on reflective surface CD and its vertex at the point of radiation. Conversely, at any random point C, H, or D on the reflective surface, a cone of

radiation is formed with its base on the luminous surface and its vertex at the point of reflection. If some point X is taken between the two surfaces, then the light projected from AB through that point to CD will form two cones with shared vertex X and respective bases AB and CH. Likewise, if some intermediate point Y is taken between X and luminous surface AB, then the rays passing through that point from CD to AB will form two cones with shared vertex Y and respective bases CD and EF. Given that reflection takes place at equal angles, moreover, each such cone will generate an equivalent cone after reflection, so that, for instance, cone of radiation AHB will generate equivalent cone A'HB' after reflection.

⁶⁵Accordingly, if EGHF in figure 4.3.19, p. 203, represents one of the circular holes in the ring, AB the luminous source, and CD the plane mirror inside the ring, then the light from AB will be projected to XY on the mirror. If some point is chosen on line GH within the circle at the bottom opening of the hole, then that point will form the vertex of two cones of radiation with their bases on the lighter circles inside AB and XY, so XY forms the spot on the mirror mentioned earlier. Ultimately, the aggregate of rays reaching through the hole from AB to the mirror will form section GXYH, which, in reflecting from the mirror, forms section KXYL. Hence, the light that spreads out from the bottom opening of the hole to the mirror continues to spread out commensurately after reflection, as was empirically determined earlier when the circle of light on the mirror was compared to the circle of light at the hole's opening and to the circle of light projected to the corresponding hole in the ring.

⁶⁶In fact, to this point in the Latin version of *De aspectibus*, Alhacen has not established these two points explicitly, although he has done so implicitly. The first point—i.e., that light-intensity weakens with distance—follows from the fact that luminous sources appear fainter as they get farther from the viewer. The second point—i.e., that concentrated light is more intense than dispersed light—follows from the fact that light radiates in all possible directions from any luminous or illuminated surface, so that the farther it gets from its source, the more it will spread out. Consequently, if an opaque body is placed in the way of that light, the closer that body lies to the source of illumination, the more light will strike its surface and, consequently, the more intensely illuminated it will be. Alhacen makes these two points explicitly in chapter 3 of the first book in the Arabic version of the *De aspectibus*; see esp. I, 3.79; I, 3.82; I, 3.121, in Sabra, *Optics*, vol. 1, 36-37 and 45.

⁶⁷The concentration to which Alhacen adverts here is due to the fact that every point on the luminous surface radiates its form to each point on the reflecting surface, so that the illumination at that point is the sum of the illumination reaching it from every point on the luminous surface.

⁶⁸How reflected light is weakened by distance and dispersal is illustrated in figure 4.3.19, p. 203, where the light shining through opening GH is projected to XY, which is larger than GH, and reflected thence to KL, which is larger than XY. Hence, the total distance the light travels in passing from GH to XY and then to KL is greater than the distance it travels in going from GH to XY, so the light is weakened accordingly. Also, as it goes from GH to KL the area upon which it is projected increases commensurately with the increase in distance, so the same amount

of light is dispersed over a greater area. And, finally, the reflection itself weakens the light. Therefore, by the time it reaches KL, the light is triply weakened. On the other hand, if KL is taken as the luminous source, then the radiation from it that reaches XY on the mirror is concentrated on a smaller area, and when it reflects to GH it is even more concentrated. Hence, if the light is intensified by that concentration enough to counterbalance the weakening due to distance and the reflection itself, the light at GH will be as intense as the light at KL. But if that intensification is not enough to counterbalance the weakening, the light at GH will be weaker than the light at KL, whereas if the intensification overbalances the weakening, the light at GH will be more intense than the light at KL.

⁶⁹Alhacen's point here can be illustrated by reference to figure 4.3.19, p. 203. Let KL represent the luminous surface, and let KXYL represent a segment of a cone of radiation emanating from KL and projected upon area XY of the mirror. Hence, the light at XY will be concentrated, since XY is smaller than KL. Now, if the illumination at XY is more intense than that at KL, then the light at any point on XY will be more intense than the light at any point on KL. Likewise, if the illumination at XY is less intense than that at KL, the light at any point on XY will be less intense than the light at any point on KL. In this case, therefore, even though the light from every point on KL is concentrated at any given point on XY according to the cone of radiation that has its base in the luminous surface and its vertex at that point, the light concentrated at that point will still be less intense than the light at any given point on KL.

⁷⁰Overall, the point of this discussion is that the light shining from any luminous source onto a mirror reaches that mirror in three ways: according to a multitude of cones with their bases on the mirror and their vertices at points on the luminous surface; according to a multitude of cones with their base on the luminous surface and their vertices at points on the mirror; and according to parallel rays connecting points on the luminous surface with corresponding points on the mirror. In the aggregate, all three types of radiation affect the intensity of reflected light through concentration, dispersal, and increasing distance. In fact, Alhacen was well aware that, despite its weakening through distance and reflection itself, parallel light-radiation can be concentrated to great intensity by being focused in concave spherical and parabolic mirrors; see note 94, p. cii above.

⁷¹It is interesting to note that Alhacen's definition of a minimal quantum of light is contingent on visibility: that is, the smallest effective quantum of light is what is minimally effective in causing sight. This seems to mean that any given luminous surface can be subdivided into smaller and smaller areas until, finally, the subdivision is so small that it can no longer be seen. From where? Alhacen does not specify, but one can suppose that it becomes invisible from a point infinitesimally beyond the surface. Likewise, one can suppose that the standard of visibility is determined by what can be seen by a healthy, normal eye. Furthermore, it seems clear that the size of such spots varies with the intensity of the luminous source, the minimal quantum of effective light on the surface of a brightly luminous source being smaller than that on the surface of a faintly luminous source. By the same token, the size of such spots will vary with the distance between source

and viewer; the farther that distance, the larger the spot. All in all, then, Alhacen's definition of least light is a function more of perceptual psychology than of physics.

⁷²That light does not actually radiate from mathematical points follows from the fact that it necessarily inheres in the surfaces of bodies. Being continuous, surfaces are not composed of points, because, by book 1, definition 1 of Euclid's *Elements*, points have no dimension whatever. Hence, no matter how far we subdivide a luminous or illuminated surface, and thus the light on it, such subdivision will never end in points. Reduced to its effective minimum, light occupies a tiny area or spot on the luminous or illuminated surface, that spot being represented mathematically by the point at its center. The light within this spot propagates its form along a shaft bounded by the edges of the spot it occupies on the luminous or illuminated surface so that when this form reaches another body, it will occupy a spot of the same size on the surface of that body. The mathematical line centered within that shaft and connecting the centerpoints of both spots of light will constitute the ray. Hence, the points at ends of the rays are virtual representations of the real spots of light centered on them, and the ray is a virtual representation of the real shaft of radiation centered on it.

⁷³For Alhacen's discussion of the swiftness of light-radiation, see 2, 3.60-62, in Smith, *Alhacen's Theory*, 445-47. At this point, Alhacen is transforming the minima of "least light" just discussed into virtual quanta so as to treat light-radiation by analogy to the projection of tiny bodies that interact dynamically with the physical objects against which they are projected. For some discussion of the source of this analogy, see Smith, *Ptolemy's Theory*, 37-38 and 42-43.

⁷⁴Here Alhacen makes clear that light is to be understood as a virtual rather than a real body in motion. Hence, as he stipulates, the activity of light in its propagation, absorption, reflection, and refraction is not due to the actual physical properties of the bodies with which it interacts, although its activity can be likened to that of bodies in motion for the purposes of descriptive "explanation." Thus, although it is not due to actual physical rebound, reflection is like physical rebound in certain fundamental ways and therefore follows its laws in certain fundamental ways.

⁷⁵By now, Alhacen's attempt to describe the activity of light by analogy to projectile motion has forced him into a quandary. On the one hand, he is quick to deny that light-radiation involves actual matter in motion. Yet, on the other, he ties reflection to the physical structure of the surfaces upon which it impinges. Moreover, his account of why light does not reflect from uneven or porous surfaces implies that, when light strikes such surfaces, its rebound is somehow deadened, causing the light to be trapped in those pores. To add even more difficulty to this account, he points out that, even if various parts of the surface are reflective but not arranged uniformly, the light that strikes them will reflect, but in various directions so as to be scattered randomly. In that case, presumably, the light that is seen comes from all parts of the body, not just a privileged spot, as happens in true reflection. This model of deadening and/or scattering, which is apparently meant to explain

mere illumination, is apparently inconsistent with his previous explanation of illumination according to the absorption of luminosity by opaque surfaces, which become sources of luminosity by virtue of that absorption.

⁷⁶The distinction between natural and accidental motion is based on the Aristotelian theory of motion according to which heavy objects, by their very nature as heavy, are intrinsically impelled to move straight toward the center of the universe in free fall. Accidental motion, being extrinsically forced, counters the natural tendency of heavy bodies to drop straight down, causing them to move in other directions, e.g., along the horizontal or straight upward. See esp. Aristotle, *Physics*, 8, 4.254b7-256a3.

⁷⁷In this case, then, the arrow is shot in the same vertical plane as it was when launched along the horizontal, but it is shot along a slanted trajectory either down toward or up toward the mirror, in which case it rebounds back at a slant.

⁷⁸Thus, the resistance posed by the body is a function of the smoothness or continuity of its surface rather than of its physical hardness. This, presumably, is why water can be intensely reflective despite its lack of physical hardness.

⁷⁹Alhacen's analysis of the dynamics of reflection is illustrated by figure 4.3.20, p. 203, in which AB represents the reflecting surface, XY the normal, R the point of reflection, and CR an oblique ray along which light reaches point R. Now, if there were no resistance whatever at R, the light would pass through surface AB unhampered, and it would continue along the rectilinear continuation RD of its original path, so the resulting path below surface AB would have precisely the same disposition as the path above it. On the other hand, if the light is completely resisted at R, it will rebound along RE, which has the same disposition as CR insofar as the angle of reflection XRE is equal to the angle of incidence CRX. The "dynamic" explanation for this follows from Alhacen's reduction of the motion of the light along CR to two components: motion along the normal XY and motion along the orthogonal to that normal, BR. Accordingly, there are two components of resistance at point R, one upward along normal RY, the other along orthogonal AB. But since the light does not pass into the mirror through R, the resistance along AR does not come into play, leaving the light to maintain its original motion in the horizontal direction. On the other hand, there is full resistance along normal RY, so the light is diverted upward with the same amount of motion as before but in the opposite direction. Later, in book 7 of the *De aspectibus*, Alhacen applies the same vectorial analysis to refraction in order to explain why, when it enters an optically dense medium, light is forced toward the normal after passing through point R, now taken as a point of refraction rather than of reflection. For some discussion of this vectorial analysis and its implications, see A. I. Sabra, *Theories of Light from Descartes to Newton* (London: Oldbourne, 1967; reprint, Cambridge University Press, 1981), esp. 72-82 and 93-98; David C. Lindberg, "The Cause of Refraction in Medieval Optics," *British Journal for the History of Science* 4 (1969): 23-38; and A. Mark Smith, *Descartes's Theory of Light and Refraction: A Discourse on Method*, Transactions of the American Philosophical Society 77.3 (Philadelphia: APS Press, 1987).

⁸⁰Presumably, then, even though light loses intensity/speed with distance as well as with the impact of reflection, its overall speed of propagation is so great

that it will not decelerate to the point of stopping no matter how far it goes or how many reflections it undergoes.

⁸¹My interpretation of this experiment is illustrated in figure 4.3.21, p. 204. The apparatus is set toward the window in such a way that the holes in line with RF and RE are blocked from any illumination passing through the window, presumably to keep such stray illumination from interfering with visual observation through those holes. The colored object is placed in direct sunlight streaming through the window so as to be brightly illuminated. Beams of luminous color will pass from C and D through the two holes in line with CR and DR to reach point of reflection R on the mirror inserted into the apparatus. Those beams will then be reflected from R to their corresponding holes, CR to the hole in line with RF and DR to the hole in line with ER. Since the illumination on the face of colored object CD is secondary, then its radiation to the mirror will be relatively weak, and it will be even weaker after having passed into the hollow of the ring through the narrow holes. Subject to additional weakening by reflection at R, the luminous color passed along RF and RE will be so faint by the time it reaches the inside wall of the ring at the openings of the holes in line with RF and RE that it may be too faint to see, even if it shines on white parchment placed at those openings. Hence, in order to verify that color reflects at equal angles, the experimenter must actually look through the holes to see the reflected color. The problem with this procedure is that it provides only indirect verification, because what is seen is the image behind the mirror of the color at the openings in line with CR and DR. However, if the hole along CR is blocked while the one in line with DR is left open, then the color's image will be seen through the hole in line with RE, but not through the hole in line with FR. And the same will hold *mutatis mutandis* if the hole in line with DR is blocked while the one in line with CR is left open.

⁸²Alhacen has already discussed this theory in the context of direct vision, imputing it to the "mathematicians," a group that certainly includes Euclid and Ptolemy. As it is articulated here, Alhacen's understanding of how mirror-images are perceived according to visual rays is unclear. He could be taken to mean that the form is grasped at the object itself, after reflection, or he could be taken to mean that the form is grasped at the mirror. This latter interpretation, however, is inconsistent with his understanding of the theory in 1, 6.58, Smith, *Alhacen's Theory*, 373-374, where he imputes to the visual-ray theorists the idea that, after making contact with a given visible object, the rays transmit the information gathered from that contact back to the eye.

⁸³In other words, the form impressed on the mirror is essentially the same as the form that would be impressed directly on the eye. Presumably, Alhacen is referring here to the Democritean theory of "emphasis," in which reflection involves a stamping of the image on the reflecting surface; see, e.g., A. Mark Smith, *Ptolemy and the Foundations of Ancient Mathematical Optics*, Transactions of the American Philosophical Society 89.3 (Philadelphia: APS Press, 1999), 34-35.

⁸⁴I take Alhacen to mean that we should block up all the holes on one side or other of the central hole in the ring that is in line with the radius of bisection on the inserted bronze plaque.

⁸⁵This brief account of how the letter would appear is confusing, in part, because the text, as it stands, makes no distinction among the curved mirrors (i.e., cylindrical, conical, and spherical) according to whether they are convex or concave. In the case of convexity, no matter whether the mirror is cylindrical, conical, or spherical, the image of the letter will appear somewhat smaller than it did in the plane mirror. Likewise, it will be bowed according to the mirror's curvature, and in the convex conical mirror it will appear more bowed toward the top according to the convergence of the lines of longitude as they approach the vertex. As in the case of the plane mirror, however, the letter will appear reversed. In the case of the concave cylindrical and conical mirrors, on the other hand, the image will appear with its proper left to right orientation, whereas in the case of the concave spherical mirror it will appear inverted.

⁸⁶In the first case, when the mirror is held sideways so that its axis lies along the horizontal, the viewer's face will appear frontal but compressed along the vertical. As the mirror is rotated upright toward the vertical, however, his face will still appear frontal, but it will become increasingly elongated until it is maximally distended at the vertical.

⁸⁷It is worth noting that none of the empirical demonstrations Alhacen adduces in this brief chapter militates against the visual-ray theory, at least not as it is applied to the analysis of reflection by Euclid and Ptolemy. Mathematically, in fact, Alhacen's theory of mirror-imaging is based on precisely the same principles as that of Euclid and Ptolemy, neither of whom subscribed to the impression-theory. Alhacen does address the visual-ray explanation of reflection later in paragraph 5.3.

⁸⁸In other words, if Z in figure 4.3.19, p. 203, were a point of luminosity on a visible object, and if KL were a visible object facing mirror CD, then the light from Z would radiate to the mirror in the form of a cone with base XY in the mirror. After reflection, that cone would be extended to base KL so that every ray originating from point Z would be reflected to a given point on KL. By extension, then, if KL were taken as a luminous object, the situation would be reversed, each point on KL radiating to base XY on the mirror and reflecting thence to Z, where all the rays would intersect.

⁸⁹The location Alhacen has in mind here is the point of reflection, not the actual location where the image appears behind or in front of the mirror. What Alhacen establishes here is that the image is always seen through, and therefore in line with, the point of reflection along the line of reflection—a point that follows from his earlier demonstrations at the end of chapter 3 and in chapter 4 that the reflected color or the image of the letter can only be seen through the corresponding hole, which lies along the line of the bronze plaque that forms an angle with the normal equal to the angle formed by the line of incidence with that same normal.

⁹⁰Here, Alhacen takes up the thread left dangling at the end of chapter 4 by explaining why the visual-ray analysis of reflection is flawed. The objection he raises here echoes his general argument against the visual ray theory in 1, 6.58, in Smith, *Alhacen's Theory*, 373-374.

⁹¹I take this to mean that, in any given instance, there is a perfect point-to-point correspondence between the entire form on the mirror and the luminous object, as

illustrated in figure 4.5.1, p. 204, where the light from points A, B, C, D, and E radiates to corresponding points F, G, H, K, and L so as to reflect to center of sight O. If the points on the entire form on the mirror were distributed disparately, as represented by ray BX, then the resulting reflection from that point would be to Z rather than O. Thus, in any given instance, the entire form of AE will be projected uniformly, so that the form is in perfect correspondence with the object. Only then will it be projected to the appropriate center of sight. Or, to put it another way, any given form at one particular spot on the mirror will be reflected to one, and only one, specific center of sight.

⁹²The intent of this passage is far from clear, but there are at least three possible interpretations. First, what might be intended is that the shape of the mirror, in terms of its contour, should be the same as that of the object in cross-section, so that a square object will only appear square in a square mirror. This, of course, is untenable, because the contour of the mirror (e.g., a plane mirror that is circular in outline) has no effect on the shape of the image seen in it. Second, Alhacen might have in mind the surface-shape of the mirror in terms of uniformity. Accordingly, if the mirror's surface is uneven, the image seen in it will also be uneven. Third, and most likely, Alhacen has in mind the distortion due to the curvature of the mirror. Hence, the image seen in a plane mirror will be the proper complement of the object in terms of apparent shape and size. In convex mirrors, on the other hand, the image will bow according to the mirror's curvature, and it generally appears smaller than it should. And, finally, in concave mirrors the image can be distorted in terms of size and orientation.

⁹³That is, according to variations in the shape of the mirror, depending on whether it is convex or concave.

⁹⁴Thus, as illustrated in figure 4.5.2, p. 205, BC represents the plane mirror itself and AB a plane continuous with its reflecting surface. R is a point on the mirror to which a ray reaches from some object-point H, and E is the center of sight. Hence, from point R a normal can be erected to the plane of the mirror, and a normal can be dropped to that same plane from E. Being parallel, those two normals form a plane. If, then, line ER is drawn from upper to lower endpoints of those normals, it will form acute angles FER and GRE and will therefore lie in the same plane as those normals.

⁹⁵Again, by reference to figure 4.5.2, p. 205, Alhacen instructs us to continue line FR on the mirror's surface to C, that line being orthogonal to both normals. Then, from right angle GRC acute angle HRG is cut so as to be equal to angle ERG, the former thus constituting the angle of incidence and the latter the angle of reflection. Accordingly, both rays and both normals will lie in the same plane of reflection, and within that plane four cardinal points are situated: center of sight E, object-point H, endpoint F of the normal dropped from E, and point of reflection R.

⁹⁶In other words, if the actual mirror is only a section of a sphere, the eye cannot lie on the mirror itself or on the surface of the sphere of which the mirror is a section.

⁹⁷Alhacen's point here is illustrated in figure 4.5.3, p. 205. Let the circle represent a section of a spherical mirror facing center of sight E, and let ET be tangent to the mirror's surface at T. Then, if ET is rotated about axis EC, which passes through

the mirror's center, T will sweep about the surface of the mirror to form a circular section represented by line TT' , according to which the circle is viewed edgewise. Thus, axis EC passes through centerpoint B of that circle, BE forming the axis or pole of that circle's generation. Now, we know from Euclid, III.18, that any line drawn from center C of any great circle within the sphere to any point of tangency, such as T, of that great circle, will be perpendicular to the tangent, so angle ETC is a right angle. Hence, the angle formed by tangent ET with pole BE of circle TT' generated by that tangent will be less than a right angle. If the great circle is then subdivided into quadrants by diameters AC and DC, it is obvious that arc AT cut off by the tangent will be less than a quadrant of that great circle. According to this analysis, then, if the mirror is convex, segment TAT' , which is less than half the sphere, will be visible. If it is concave, on the other hand, segment TDT' , which is greater than half the sphere, will be visible, assuming that convex segment TAT' is excised to leave the entire section open to view.

⁹⁸Alhacen's demonstration that any point within the visible portion of a spherical mirror can be a point of reflection is illustrated in figure 4.5.4, p. 206. The elliptical section $TAT'B$ backgrounded in light gray in the upper diagram represents the section of a convex spherical mirror that is visible from center of sight E, and EPC, passing through the sphere's centerpoint C, represents the pole or axis of that section, point P being where it intersects the sphere's surface. $TPRT'$ is the arc of a great circle on the spherical surface of the mirror and it passes through point P and some randomly chosen point R. Line DRF is drawn tangent to that arc at R. ARB is the arc of another great circle passing through R, GRK being tangent to that arc at point R. Therefore two tangents DRF and GRK form a plane tangent to the sphere at R, so line XRC is perpendicular to that plane and thus normal to the mirror at point R. Likewise, axis EPC of the visible section of the mirror is normal to the mirror since it passes to C. Accordingly, since EPC and XRC are both normal to the mirror, they lie in the same plane, which is perpendicular to the plane formed by tangents DRF and GRK. Within the plane containing EPC and XRC, as represented in the lower diagram of figure 4.5.4, line of reflection ER strikes the mirror obliquely at angle ERX, so some line of incidence OR can be dropped to R to form angle of incidence ORX equal to ERX. Therefore, R is a point of reflection, and the same will hold for any point on arc TT' . Since an infinite number of planes can be passed along EPC to cut great circles on the mirror, and since every point on each of those great circles can serve as a point of reflection for center of sight E, then every point on visible portion $TAT'B$ of the mirror can serve as a point of reflection for E.

⁹⁹In other words, given viewpoint E, all possible planes of reflection will intersect along normal EPC in figure 4.5.4, p. 206, and those planes will cut great circles on the surface of the sphere.

¹⁰⁰Alhacen's point here is illustrated in the top diagram in figure 4.5.5, p. 207. Let convex cylindrical mirror $T'ABT$ face center of sight E, let the cylinder be cut by plane ETT' , passing through E, and let the resulting common section be circle TT' centered on C. Draw tangents ET' and ET to that circle, and then from points T and T' drop lines of longitude $T'A$ and TB. Neither of the two planes formed by tangent ET' and line of longitude $T'A$ or by tangent ET and line of longitude TB will cut the

cylinder, because if it did—e.g., if plane ETB were to cut the cylinder—then that plane would cut the cylinder along two lines of longitude, TB and some other line of longitude, such as LM. But, in that case, tangent ET would cut the circle at two points, T and L, so it would not be tangent, which is contrary to our original supposition. Therefore, the two planes ET'A and ETB are tangent to the mirror, and the portion of the convex mirror visible to the eye at E is defined by lines of longitude T'A and TB. Although Alhacen specifies at the beginning of this construction that the mirror be convex, it is clear in this case, as in the case of the spherical mirror, that lines of longitude T'A and TB also define the maximum visible section of the concave surface of the cylinder facing viewpoint E, provided that convex segment T'PTBA is excised to expose that entire section to view.

¹⁰¹In other words, as illustrated in the top diagram of figure 4.5.5, p. 207, since common section ES of the two tangent planes is perpendicular to the plane of the circle cut on the cylinder, and since axis CQ of the cylinder, as well as lines of longitude T'A and TB, is also perpendicular to that plane, then ES, CQ, T'A, and TB are all parallel to one another.

¹⁰²That reflection can occur to E from any point on the visible portion of the convex cylindrical mirror is illustrated in the middle and bottom diagrams of figure 4.5.5, p. 207. Let E lie in the plane of the cutting circle TT' centered on C, as illustrated in the middle diagram. If the point selected is P, through which the visual axis passes to centerpoint C of the circle, then reflection will occur back along PE, since that line is normal to the mirror. If R is selected to the side of visual axis EP, then a line of longitude RU will pass through that point. Accordingly, plane DFHG tangent to the mirror along that line of longitude will cut both tangent planes between the mirror's surface and center of sight E. The same will hold for any plane tangent along any line of longitude within the visible portion of the mirror. Now, if normal XRC is drawn to plane DFHG, then line ER connecting the center of sight and the point of reflection will form an acute angle, ERX, with that normal, since it lies within right angle DRX. Therefore, within the adjacent right angle XRF, an acute angle XRO can be cut off equal to acute angle ERX. Since all three lines ER, XR, and RO lie in plane ETT' containing the center of sight and the circular section on the mirror, ETT' will constitute the plane of reflection.

On the other hand, if R does not lie in plane ETT', then let it be chosen below that plane, as illustrated in the bottom diagram of figure 4.5.5. If it lies on line of longitude PRU passing through point P, where visual axis EC intersects the mirror's surface in plane ETT', then reflection can occur from any point, such as R, on that line, since we established earlier that ES, which is the common section of the tangent planes, and axis CQ are parallel, and line of longitude PRU is parallel to both. Hence, since EPC lies in the same plane as those three parallels, and since it is normal to the mirror, the plane within which it lies is also normal to the mirror. Within that plane any point on line of longitude PRU will be a point of reflection, for, if we pass a circle of section through that point, extended radius C'RX of that circle will be normal to line of longitude PRU. ER, connecting the center of sight with the selected point of reflection, will therefore form an acute angle with normal XRC', and within the plane of that angle an equal acute angle XRO can be formed

with normal XRC' on the other side. The resulting line of incidence, OR , will therefore lie in the same plane as normal XRC' and line of reflection ER , and that plane is normal to the mirror. If we choose another point of reflection, such as R' , to the side of PRU , then we can draw a line of longitude $R'U$ through it, and along that line of longitude we can form tangent plane $DFHG$. We can then form the circle of section through point R' , diameter $C'R'X'$ of that circle being normal to plane $DFHG$ and thus to the mirror. Hence, line ER' connecting the center of sight and the point of reflection will form acute angle $ER'X'$ with normal $X'R'C'$, and within the same plane an equal acute angle $X'R'O'$ can be formed with normal $X'R'C'$. Since the plane formed by line of incidence $O'R'$, normal $X'R'C'$, and line of reflection ER' is normal to the mirror at point R' , that plane will be a plane of reflection.

As with the spherical mirror, so with the cylindrical mirror, the analysis holds for the concave section of the mirror bounded by $T'A$ and TB if we extend normals XRC' and $X'R'C'$ to points Z and Z' on that surface. Since those lines are normal to the concave surface of the cylinder at those points, then lines EZ and EZ' will form acute angles with the tangents to those points. Within the plane of each of these angles an equal acute angle can be formed with that tangent on the other side of the normal. Since the plane formed by each line of incidence, its respective normal, and its respective line of reflection will be normal to the mirror, it will be a plane of reflection.

¹⁰³That the common section of a cylinder and a plane cutting the cylinder obliquely is a true ellipse was first proved by Serenus in proposition 20 of *On the Section of a Cylinder*.

¹⁰⁴If the plane of reflection forms an elliptical section on the cylinder's surface, as represented in the top diagram of figure 4.5.6, p. 208, by the ellipse, whose major axis is FG , line FG will be bisected at C , where it intersects axis TU of the cylinder. Pass a plane through point C orthogonal to axis TU to form circle AB . C will therefore be the circle's centerpoint, and its diameter RCZ will constitute the common section of the circle's juncture with the ellipse. Since it coincides with the diameter of circle AB , XRC is normal to the plane tangent to the cylinder's surface at point R along line of longitude DY , and it is also perpendicular to axis TU . Thus, for any center of sight stationed within the plane of the ellipse, reflection will occur to it from point R , provided that R lies within the visible portion of the mirror. On the other hand, reflection cannot occur from any other point, such as L , on the elliptical section, because line LC dropped from it to C will fall obliquely to axis TU and will therefore not be normal to the mirror's surface. Or, to put it another way, line LS normal to the ellipse at point L will pass by axis TU and will therefore not intersect it.

In sum, then, within the plane of reflection formed by the ellipse, the only point from which reflection can occur is that point on the mirror where the ellipse is intersected by a circle, such as AB , whose diameter forms the common section of the circle and the ellipse, this section also forming the minor axis of the ellipse. Being shared by the circle and the ellipse, this is the only line within the ellipse that is perpendicular to axis TU , so it is the only line within the plane of the ellipse that is normal to the mirror's surface. It should be noted that the same analysis applies

to the concave surface of the mirror beyond C. Thus, if normal XRC is extended to point Z on the opposite side of the cylinder, point Z will be a point of reflection for any center of sight within the plane of the ellipse. In this case, however, opposite point R can also serve as a point of reflection for that center of sight if more than half the mirror is visible to it. Alhacen makes this point later on in paragraph 5.56 below.

At first glance, the conclusion that no point other than R will serve as a point of reflection in the plane of the ellipse seems to contradict the earlier claim that, for any given center of sight, every point in the visible portion of the cylinder will be a point of reflection. Thus, it seems inconsistent to deny that L can be a point of reflection for the center of sight for which R is a point of reflection. In fact, it can, but not within the plane of the given ellipse. Thus, if E in the bottom diagram of figure 4.5.6 represents a given center of sight, then, as was just shown, R is the only point of reflection for E within the plane of elliptical section RL, because only that section shares diameter XRC of the circle of section ARM passing through R. Accordingly, XRC will be normal to the mirror along line of longitude DR, and line of reflection ER and normal XRC will lie in the plane of elliptical section RL. Now, if we pass circle of section PLU through point L, then its diameter X'L will be normal to the cylinder along line of longitude VL. If line EL is drawn from the center of sight to point L, then that line will form acute angle ELX' with normal X'L. It will also form a plane with that normal, and within this plane an acute angle equal to ELX' can be formed on the other side of the normal. Thus, all three lines will lie in a plane that is normal to the mirror at point L. Moreover, that plane will cut an elliptical section on the cylinder, in this case the section passing through points N and L. Thus, points E and L lie in both of the planes of both elliptical sections, the original one passing through points R and L, and the new one passing through points N and L. And the same holds for any other point on the original section passing through points R and L: another plane of reflection can be passed through it from E according to the diameter of the circle of section at that point, since that diameter is normal to the cylinder at that point.

¹⁰⁵In other words, more than half the surface of a cylindrical mirror is visible to both eyes if the diameter of the cylinder's base is less than the distance between the eyes. In that case, if both eyes lie within the plane of the ellipse, two points of reflection may be exposed to it, because diameter RC of the circle, which is also the minor axis of the ellipse, is normal to the mirror's surface at both its endpoints.

¹⁰⁶This, of course, is simply a repetition of the argument made with respect to convex cylindrical mirrors in paragraph 5.16 above.

¹⁰⁷The situation described to this point is illustrated in figure 4.5.7, p. 209, where the cone with vertex A stands on base-circle GF, which is orthogonal to axis AC passing through centerpoint C of that base-circle. Center of sight E is located below A so that line EA connecting them forms acute angle EAC with axis AC. A plane is passed through E orthogonal to axis AC to cut the cone along circle DB. Tangents EB and ED are drawn to that circle, and through points of tangency B and D lines of longitude AF and AG are drawn. The two planes formed by lines of tangency EB and ED and by lines of longitude AF and AG, respectively, are tangent

to the surface of the cone along the lines of longitude, neither of them being able to cut the cone. Therefore, the lines of longitude AF and AG along which these two planes are tangent to the cone will define the visible portion of the mirror, which constitutes less than half its surface. As in the case of the concave spherical and cylindrical mirrors, so in this one, if the convex segment AGF of the mirror facing the eye were removed, the concave portion defined by lines of longitude AG and AF would be visible to center of sight E, and it would constitute more than half the cone's surface.

¹⁰⁸In figure 4.5.8, p. 209, center of sight E faces the right cone so that line EA drawn from it to the cone's vertex forms a right angle with axis AC. A plane is passed through the cone orthogonal to axis AC to form circle DKB passing through point H on the axis. H is therefore the circle's centerpoint. The common section of this circle and plane EAHK containing the center of sight and axis AC is straight line KH, which is a radius of circle DKB and is therefore perpendicular to axis AC. Diameter DHB is drawn orthogonally to KH, and through endpoints B and D of that diameter tangents LB and MD are drawn to the circle. From points of tangency B and D lines of longitude BA and DA are drawn to the cone's vertex. Hence, the planes formed by tangent LB and line of longitude AB and by tangent MD and line of longitude AD are tangent to the surface of the cone along those lines of longitude. Since EA and KH lie in the same plane, and since both lines are perpendicular to axis AC, they are parallel to one another. Likewise, tangents MD and LB are parallel to one another as well as to EA and KH, so planes EABL and EADM intersect along EA. Accordingly, the visible portion of the cone defined by planes EABL and EADM constitutes half the cone's surface, since the defining planes are tangent at endpoints D and B of the circular section's diameter (as well as at endpoints F and G of the base circle's diameter). As before, if the visible concave segment of the mirror were removed, the concave segment defined by lines of longitude AF and AG facing the mirror would be visible, and it would constitute half the mirror's concave surface.

¹⁰⁹The situation just described is illustrated in figure 4.5.9, p. 210. Center of sight E faces the conical mirror from above so that line EA drawn from it to the cone's vertex forms obtuse angle EAC with axis AC. A plane is passed through the cone orthogonal to axis AC to form circle DB. The common section of this circle and plane EAC containing the center of sight and axis AC is diameter KH of the circle. HK is extended beyond the cone's surface, and EA is likewise extended to meet it at point L, EAL and HKL thus forming acute angle ALK and also forming acute angle LAH with axis AC. From L tangents LB and LD are drawn to the circle, and lines of longitude AF and AG are drawn through points B and D of tangency. Thus, the planes formed by tangent LB and line of longitude AF and by tangent LD and line of longitude AG are tangent to the surface of the mirror along those lines of longitude. Those planes also intersect along line EAL, so each contains the center of sight, the vertex of the cone, and the point of intersection L. Hence, the portion of the cone's surface enclosed by the two planes on the side of L is less than half, so the portion seen by E on the other side of L will be more than half. This, of course, means that, if concave segment AFG of the mirror on the side of L were

open to view, it would constitute the maximum section of the concave surface visible to center of sight E, and it would be less than half the mirror's surface.

¹¹⁰The point of this discussion is illustrated in figure 4.5.10, p. 210, where center of sight E faces the cone from above so that line of sight EA connecting the center of sight and the vertex of the cone coincides with line of longitude AF on the cone's surface. Accordingly, no matter what point is chosen on that surface, as long as it does not lie on AF, the line from E to it will cut the surface of the cone according to the plane formed by the line of longitude on which that point lies and line of longitude AF. Thus, if points B and D are chosen on the mirror's surface, their respective cutting planes will be ABF and ADF, within which will lie triangles EAB and EAD formed by lines EB and ED from the center of sight to the object-points, the lines of longitude BA and DA extended to vertex A from those object-points, and line EA from the center of sight to the cone's vertex. Accordingly, given any such point, e.g., D, the line extended to it from center of sight E will cut the cone at no other point, such as X, so the line of sight to D will be uninterrupted. It follows, then, that all the points on the mirror's surface, except for those along line of longitude AF, will be exposed to view. On the other hand, if some concave portion of the mirror were open in this situation, for instance, if segment ABD were excised, none of its surface except for line of longitude AF would be visible, because, other than EA, none of the lines drawn from E to any point on its interior surface would cut it. In short, the entire concave surface would be effectively invisible.

¹¹¹Alhacen's point here is illustrated in figure 4.5.11, p. 211, where line of sight EAF enters the cone through vertex A. Accordingly, any line of sight from E to any point on the mirror will cut the mirror on the line of longitude passing through that point. Thus, the plane tangent to the mirror along that line of longitude will pass below the center of sight, and since the plane containing the center of sight, the line of sight, and the line of longitude is orthogonal to that tangent plane, it can be a plane of reflection. Conversely, if any segment of the mirror is removed so as to open the concave surface of the mirror to view, none of that surface will be visible, since none of the lines of sight will intersect that surface.

¹¹²Although this last sentence is somewhat tortured in Latin, the point of it seems to be that, no matter where the line of sight touches the cone's surface within the visible portion of the mirror, that line of sight lies in a plane that is orthogonal to the plane tangent to the mirror at that point, so not only does that former plane constitute a plane of reflection—or, as Alhacen calls it here, a visual plane (*visualis superficies*)—but the point of tangency is visible. Accordingly, at least by implication, that point of tangency will be a point of reflection not just in theory but in fact.

¹¹³The claim that reflection will occur from any point on the visible surface of the cone can be easily understood by recourse to figure 4.5.12, p. 211. Let R be a randomly chosen point on the surface of the cone, whose vertex is A and whose axis is AC. If a plane is passed through R and axis AC, it will cut the cone's surface along line of longitude AB. Pass a plane through R orthogonal to axis AC to cut the cone's surface along circle FRG, whose diameter CR is the common section of plane ARC and the plane of the circle. CR intersects line of longitude AB obliquely so as to form acute angle DRB with it. Let the center of sight lie outside the cone within

plane ARC, and draw tangent XY to point R. From point K on axis AC draw KRL normal to the plane formed by tangent XY and line of longitude AB. Accordingly, plane AKRL containing axis AC, line of longitude AB, and normal KRL will be orthogonal to the plane formed by tangent XY and line of longitude AB, so plane AKRL will form a plane within which reflection will occur from R to any center of sight lying within it. For instance, if the center of sight is located at endpoint L of normal KL, then the form of L will reflect back along RL from point R. If the center of sight is located at E or E' within plane AKRL but outside normal KL, then angles of incidence ORL and O'RL can be formed equal to angles of reflection ERL and E'RL, which means that the forms of O and O' will reflect from R to E and E' respectively. No matter where the center of sight is situated within that plane of reflection, then, R will serve as a point of reflection for it. Furthermore, since plane of reflection AKRL is orthogonal to the plane formed by AR and XY, which is tangent to the cone's surface along AR, then, if the lines of incidence and reflection OR and RE are held in place and rotated about axis KL, they will form the same angles of incidence and reflection throughout that rotation. Hence, no matter how the line of sight cuts the cone at R, it will lie in a plane orthogonal to the plane tangent to the line of longitude from point R, so R will be a point of reflection for E no matter where E is situated with respect to line of longitude AB.

¹¹⁴If the center of sight lies above the cone on its axis, then clearly the line of sight and the axis will coincide, and every plane passing along that line will be orthogonal to the circle at the cone's base and will intersect it along a diameter. Every such plane will therefore cut the cone along opposite lines of longitude that pass through the endpoints of a diameter on the circle at the base, so any given line of sight extended from the center of sight to the mirror's surface will intersect it at a point on the line of longitude lying in the same plane as the line of sight. Since that plane is orthogonal to the mirror's surface along that line of longitude, it will be a plane of reflection, and every point on that line of longitude will be a point of reflection for the center of sight situated within that plane, because a normal can be extended from the axis through that point. On the other hand, if the center of sight lies anywhere else, an infinite number of planes can be passed through it to cut the mirror along an infinite number of common sections. Of these, only one can be a line of longitude, and reflection will occur from any point on it. If the plane cuts a circle on the mirror's surface, then no reflection can occur in that plane, because every line within it will cut the cone's surface obliquely, so none of them can be normal to that surface. The rest of the planes will form parabolic, hyperbolic, or elliptical sections on the mirror's surface, depending on the obliquity of the cut. Accordingly, if the plane passes through the entire surface of the cone, the cut will be elliptical; if the plane is parallel to one of the lines of longitude, the cut will be parabolic, and if the plane does not cut the entire cone and is not parallel to any of the lines of longitude, the cut will be hyperbolic. That each of these planes *can* be a plane of reflection follows from our analysis of figure 4.5.12, p. 211, where we concluded that R will constitute a point of reflection for center of sight E throughout the rotation of the plane of reflection about normal KRL. During this rotation, that plane cuts the cone along an infinite number of sections, none of which forms a

circle, one of which forms a line of longitude (i.e., AB), and the rest of which form various conic sections depending on the obliquity of cut.

¹¹⁵Let the cone with vertex A in figure 4.5.13, p. 212, represent the mirror, and let the plane of reflection cut the mirror at point R along conic section DRE. Extend normal KRL through point R to intersect the cone's axis at K. Circle RB imagined to pass through R will intersect axis AK orthogonally, and AK will pass through its centerpoint C, so, like BCR, all its diameters will intersect the mirror's surface obliquely. If normal KRL is rotated about axis AK, it will form a cone with its vertex at K and its base on circle RB. Therefore, all the lines of longitude on this second cone will be normal to the surface of the cone forming the mirror. Let line KF be one of these lines of longitude, and let it extend through circle RB to G. It will therefore be normal to line of longitude AF on the mirror. Then, through point H, where line of longitude AF intersects conic section DRE formed by the cutting plane, extend line KHM from the vertex of the lower cone. Clearly, since KHM and KFG lie in the same plane with line of longitude AFH on the mirror, KHM must be oblique with respect to that line, since KFG is perpendicular to it. Accordingly, H cannot be a point of reflection in the plane of DRE, nor, for that matter, can any point on section DRE other than R. Thus, within plane of reflection DRE there can be only one point of reflection. Furthermore, since the conic section is tangent to the lower cone along line of longitude KR, that line will form the axis of the section. Hence, if RK intersects edge AB of the cone, the section will be an ellipse with RK as its major axis. If RK is parallel to edge AB of the cone, the section will be a parabola, and if RK does not intersect AB within the cone, the section will be a hyperbola. Which of the three conic sections is formed ultimately depends on the angle RAB formed at the vertex of the cone. In all three cases, K will be the focus of the conic section. This analysis also applies if the mirror is concave so that, instead of viewing the mirror from the left, the center of sight views it from the right. As long as it lies in the plane of conic section DRE, and as long as the major axis of that section passes through R and K, R will be the only point of reflection for that center of sight within that particular plane.

¹¹⁶That there can be two, but no more than two, points of reflection in a plane that cuts the imagined cone is illustrated in figure 4.5.14, p. 212. Let the plane be passed through line of longitude AF on the mirror's surface to cut the base-circle of the imagined cone at points R and R' and to form conic section DRR'E on the mirror. By construction we established that lines of longitude KRM and KR'L on the imagined cone are normal to lines of longitude AR and AR' on the mirror's surface. Thus, if those two lines of longitude in the imagined cone lie within the plane of conic section DRR'E, then R and R' can serve as points of reflection for any center of sight lying outside the mirror's surface within that plane. Within that same plane, moreover, lines of reflection from both R and R' can intersect at one and the same center of sight, so, if the center of sight is stationed far enough beyond the cone that both R and R' are exposed to view, reflection can occur from both points to that center of sight. That reflection can occur to that center of sight from no other points on conic section DRR'E is clear from the fact that, if any other point on the conic section is chosen, and if a line from vertex K of the lower cone is extended

through it, that line will not be normal to the section. For instance, if point P is chosen on section DRR'E below base-circle FB, and if line of longitude AP on the mirror is dropped to that point, then, when line of longitude KPQ on the lower cone is extended through it, it will form obtuse angle APQ with line of longitude AP on the mirror, because it passes below point O, where normal KO intersects the same line of longitude. On the other hand, if point H is chosen on section DRR'E above base-circle FB, and if line of longitude AF on the mirror is dropped to that point, then, when line of longitude KHN on the lower cone is extended through it, it will form acute angle AHN with AF, because it passes above point F, where normal KF intersects the same line of longitude. Therefore, no matter how the imagined cone with vertex K is cut, as long as two lines of longitude on that cone lie within the plane of that cut, reflection will occur from the two points on the resulting conic section where it intersects the base-circle of the imagined cone. Since there is an infinite number of pairs of lines of longitude within that lower cone, then there is an infinite number of possible cuts in that cone that will yield a pair of reflection-points on the circle common to it and the cone of the mirror.

It is important to note that the lower cone, and therefore its base-circle, are absolutely specific to the conic section formed by the planar cut. As we have seen, its vertex must lie at that section's focal point, and the lines of longitude on it must cut the base-circle so as to be perpendicular to the lines of longitude on the mirror where they intersect that circle. Hence, the key to grasping Alhacen's point in this case is to understand that, no matter how the plane cuts the mirror, the point where it intersects the mirror's axis is on the axis, or major axis, of the resulting conic section and lies at that section's focus. Accordingly, if we produce a cone whose vertex is at that point and whose generatrix is normal to the surface of the mirror, the base-circle of that newly generated cone will be intersected by the conic section produced on the mirror's surface by the original planar cut at at least one point, or at most two. In the case of only one point of intersection, the line of longitude on the lower cone passing through that point will be the conic section's axis or major axis, and that point of intersection will be the sole point of reflection within that plane. If, on the other hand, the base-circle is intersected at two points, then those will be the only two points from which reflection can occur within that plane. By the same token, if the mirror is concave, R and R' will be the only two possible points of reflection for a center of sight stationed to the right of line of longitude AF and within the plane of conic section DRR'E.

¹¹⁷This point has already been established in note 98 above.

¹¹⁸In figure 4.5.15, p. 213, the circle represents a section of a concave spherical mirror centered on C, E the center of sight, and R the chosen point, which does not lie on diameter EC passing through center of sight E. Draw tangent XRY at point R, and draw normal CR and line of sight ER, both of which form acute angle CRE at R. Whether E lies inside or outside the sphere, ER will fall inside the sphere since it enters through the opening between tangent lines of sight TE and T'E. Now, since normal CR forms right angle CRX with tangent XRY, and since ER lies within that right angle, an angle equal to angle CRE can be cut off from right angle CRY on the other side of CR. Let that angle be CRO. Like line of sight ER, line RO lies inside

the sphere, because angle CRO is acute, and the angle formed by the circle with CR is greater than any acute rectilinear angle (Euclid, *Elements*, III, 16). Therefore, since CRO is an acute rectilinear angle, RO cannot fall outside the circle (i.e., between the circle's outside surface and tangent XRY). According to this analysis, then, in the case of a spherical concave mirror, the center of the sphere, the center of sight, the point of reflection, the object-point, and the endpoint of the diameter extending from the center of sight to the surface of the mirror through its centerpoint will all lie in the plane of reflection, which forms a circular section on the sphere. It is worth noting that, in treating OR as a line of reflection rather than as a line of incidence, Alhacen couches his analysis in terms of visual rays rather than of light-rays. Clearly, however, he does so for the sake of analytic convenience rather than from confusion over whether vision is due to light-rays or visual rays, since he is unequivocal in his rejection of the latter.

¹¹⁹Thus, as has already been established in note 100 above, if the cylinder in the top diagram of figure 4.5.5, p. 207, represents a concave cylindrical mirror facing center of sight E, then the maximal portion of it that is visible to E outside the mirror consists of the segment between tangent planes ETBS and ET'AS on the side opposite axis KL from E.

¹²⁰This point has already been established in note 118 above. If the circle in figure 4.5.15, p. 213, is taken to represent the common section of the plane of reflection and a cylinder rather than the common section of the plane of reflection and a sphere, then everything that applies in the analysis of the sphere applies in the analysis of the cylinder.

¹²¹This point has already been established in note 105 above. Thus, as illustrated in the top diagram of figure 4.5.6, p. 208, there are two possible points of reflection on an elliptical section in a convex cylindrical mirror. Those points, R and Z, lie at the intersection of the minor axis of the ellipse and the circle passing through the endpoints of that axis. In this case, however, only one of those points is visible to a single center of sight, because the visible portion of the mirror is less than half. The visible portion of a concave cylindrical mirror is more than half, however, so both points of reflection can be seen from a single center of sight whether it lies inside or outside the mirror, as long as the opening in the mirror is wide enough to allow both lines of sight to reach those points.

¹²²As already established in notes 110 and 111 above, this follows from the fact that in the two cases described all of the lines of sight projected to the mirror will strike its outer surface, so none of them will reach its inside surface, no matter how large an opening in it is provided for the lines of sight to pass through. Thus, as illustrated in figure 4.5.10, p. 210, if line of sight EAF extends along a line of longitude on the mirror, it is clear that no line of sight from E will reach the inner surface of the mirror. This restriction will apply no matter how wide the opening in the mirror is. Likewise, if the line of sight from E to A enters the cone, as in figure 4.5.11, p. 211, then, no matter how wide the opening in the mirror, none of the lines of sight projected from E will reach the inside surface of the mirror.

¹²³This point is clear in figure 4.5.7, p. 209, where the inner portion of the cone visible through opening GAF facing center of sight E constitutes more than half the mirror. This situation applies when the line connecting the center of sight and the

cone's vertex forms an acute angle with the axis. When that angle is right, precisely half the inner surface of the mirror will be visible, as illustrated in figure 4.5.8, p. 209, where the tangents defining the visible portion of the mirror pass through the endpoints D and B of the circle's diameter. When the angle formed by line of sight EA and the axis is obtuse, less than half of the mirror's inner surface will be visible, as illustrated in figure 4.5.9, p. 210, where the maximum visible portion of the mirror is defined by concave segment AFG facing the eye.

¹²⁴In other words, when the base forms the only opening in the mirror, and the center of sight faces that opening, the only points that can be reflected to it will be points on the mirror's inner surface within the plane of reflection or points on some object placed inside the mirror.

¹²⁵This point is illustrated in figure 4.5.16, p. 214, where ABOP represents a cross-section of the truncated cone, E a center of sight inside the mirror, and E' a center of sight outside the base. The forms of D and N reflect to E such that their lines of incidence pass just over edges A and B of the opening in the mirror. Hence, nothing beyond points N and D along horizontal line ND will be visible to E by reflection. Likewise, the forms of F and M reflect to E' such that their lines of incidence pass just over edges B and A of the opening in the mirror. Hence, nothing beyond points F and M along horizontal line ND will be visible to E' by reflection. The forms of L and G, on the other hand, reflect to E from just below edges A and B of the opening in the mirror, so nothing between G and L will be visible to E by reflection. Likewise, the forms of K and H reflect to E' from just below edges A and B of the opening in the mirror, so nothing between K and H will be visible to E' by reflection. Thus, the field of reflected view for E is bounded on the outside by the largest circle, and on the inside by the next smallest, whereas the field of reflected view for E' is bounded on the outside by the next largest circle, and on the inside by the smallest. Consequently, although the internal blind spot for E' is smaller than that for E, its overall field of vision is considerably smaller. Note that Alhacen couches his discussion in terms of the scope of egress for lines of reflection (*pateat exitus lineis reflexionis . . . liberum habebunt linee gressum . . . latior reflexis lineis datur ad egrediendum*), although it is clear that he is really referring to the scope of ingress for lines of incidence. Again, he seems to be using the language of visual rays for analytic convenience rather than out of confusion over whether vision is due to visual rays or light-rays.

¹²⁶Note the sharp distinction between "form" and "image," the latter being applied to what is seen in a mirror, which is an illusion, whereas the form itself is a representation.

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[2.68-70] PROPOSITION 5: In convex spherical mirrors the image-location lies at the intersection of the normal dropped from the object-point and the line of reflection. **[2.71-73]** PROPOSITION 6: In convex spherical mirrors the line extending from the center of curvature to the image-location is longer than the line extending from the image-location to the point of reflection. **[2.74-76]** PROPOSITION 7: In convex spherical mirrors the line dropped orthogonally to the sphere from the endpoint of tangency is shorter than the radius of the mirror. **[2.77-80]** PROPOSITION 8: Along the normal dropped from the center of sight itself, only the point on the surface of the eye through which that normal passes will be seen. **[2.81-95]** PROPOSITION 9: Within the visible portion of a convex spherical mirror, some of the images appear to lie inside the sphere from which the mirror is composed, some on the surface of that sphere, and some outside that sphere. **[2.96-102]** PROPOSITION 10: When the line of sight passing through the point of reflection cuts the normal dropped from the object-point in such a way that the segment of the line of sight between the mirror's surface and the normal is equal to the segment of the normal between the point where the line of sight intersects it and the center of curvature, there can be no image-location on this latter segment. This point of intersection constitutes the limit-point of images in the mirror. **[2.103-106]** PROPOSITION 11: As long as the normal dropped from the object-point intersects the visible portion of the mirror, the image-location will lie between the previously-determined limit-point and the surface of the mirror. **[2.107-110]** PROPOSITION 12: In the invisible portion of the mirror, if the limit-point of images lies on the surface of the sphere from which the mirror is formed, all possible image-locations will lie between that point and the point where the line of sight tangent to the mirror intersects the normal dropped from the object-point. **[2.111-114]** PROPOSITION 13: Within the arc on the invisible portion of the mirror above the limit-point defined in the previous theorem, some of the image-locations will lie inside the sphere defining the mirror, some will lie outside it, and one will lie on the surface of that sphere. **[2.115-119]** PROPOSITION 14: Within the

arc on the invisible portion of the mirror below the limit-point defined in proposition 12, all possible image-locations will lie outside the sphere defining the mirror. [2.120-124] PROPOSITION 15: There is a point in the invisible portion of the mirror at which no image-location is possible. [2.125-132] PROPOSITION 16: The form of any given object-point will reflect to any given center of sight from only one point on a spherical convex mirror. [2.133-136] PROPOSITION 17: If two object-points are taken on the same normal, the image-location of the object-point nearer the center of curvature lies farther from the center of curvature than the image-location of the object-point farther from the center of curvature, and the point of reflection for the nearer object-point lies farther from the center of sight than the point of reflection for the farther object-point. [2.137-140] PROPOSITION 18: Given an object-point and a center of sight facing a convex spherical mirror and lying the same distance from the center of curvature, to find the point on the mirror's surface at which the form of the one reflects to the other. [2.141-157] PROPOSITION 19, LEMMA 1: If a diameter is taken in a circle, and if a point is taken on that circle's circumference, a line can be drawn from that point to the extension of the diameter beyond the circle such that its extension from the point where it intersects the circle to the point where it intersects the diameter is equal to a given line. [2.158-166] PROPOSITION 20, LEMMA 2: From any given point on a circle outside its diameter, a line can be drawn through the diameter to the opposite arc of the circle so that the segment on it between the diameter and that arc is equal to a given line. [2.167-173] PROPOSITION 21, LEMMA 3: Given a right triangle, and given a point on one of the sides forming the right angle, a line can be drawn from that point to the other side forming the right angle and intersecting the third side facing the right angle in such a way that the segment of the line between the point of intersection and the side on which the point does not lie is to the segment of the side opposite the right angle from the intersection-point to the side containing the given point as some given line is to another given line. [2.174-185] PROPOSITION 22, LEMMA 4: Given two points, and given a circle, to find the point on that circle such that the line tangent to the circle at that point bisects the angle formed by the lines drawn from the aforementioned two points to that tangent-point. [2.186-192] PROPOSITION 23, LEMMA 5: Given a circle, given a diameter in it, and

given some point outside it, a line can be drawn from that point to the diameter so as to cut the circle in such a way that the segment of that line from where it intersects the circle to where it intersects the diameter is equal to the segment of the diameter between where the previous line intersects it and the center of the circle. [2.193-197] PROPOSITION 24, LEMMA 6: Given a right triangle, and given a point on either of the sides containing the right angle, a line can be drawn through that point from the other side containing the right angle to the side facing the right angle such that this entire line is to the segment between the vertex of the triangle and the point where the aforementioned line intersects the side opposite the right angle as some given line is to another given line. [2.198-215] PROPOSITION 25: Given an object-point and a center of sight facing a convex spherical mirror at unequal distances from the center of curvature, to find the point on the mirror's surface at which the form of the one reflects to the other. [2.216-221] Objects seen with both eyes in convex spherical mirrors appear single even though each eye sees a slightly different image.

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[2.222-225] Preliminary observations. [2.226-230] PROPOSITION 26: If the plane of reflection cuts an elliptical section on the surface of a convex cylindrical mirror, the normal dropped from the object-point to the mirror within that plane will intersect the normal passing through the point of reflection within that same plane. [2.231-237] PROPOSITION 27: Given a point of reflection on such an elliptical section, and given a perpendicular dropped to that section to intersect the normal passing through the point of reflection, then, depending on what point on the perpendicular is chosen as object-point, the resulting image-location may lie on the section itself, and thus on the surface of the cylinder from which the mirror is composed, or it may lie inside the section and the cylinder, or it may lie outside it. [2.238-247] PROPOSITION 28: Given an object-point and a center of sight facing a convex cylindrical mirror, reflection can occur from the one to the other at only one point on the mirror's surface. [2.248-249] PROPOSITION 29: Given an object-point and a center of sight facing a convex cylindrical mirror, to find the point on the mirror's surface at which the form of the one reflects to the other.

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[2.250-253] Preliminary observations. **[2.254-268]** PROPOSITION 30: Given an object-point and a center of sight facing a convex conical mirror, reflection can occur from the one to the other at only one point on the mirror's surface. **[2.269-297]** PROPOSITION 31: Given an object-point and a center of sight facing a convex conical mirror, to find the point on the mirror's surface at which the form of the one reflects to the other. **[2.298-299]** Objects seen with both eyes in convex conical mirrors appear single even though each eye may see a slightly different image.

[2.300-490] *Mathematical Analysis of Concave Spherical Mirrors* 446

[2.300-311] PROPOSITION 32: In concave spherical mirrors the normal dropped from the object-point to the mirror will either be parallel to or will intersect the line of reflection. When it intersects, the image-location may lie on the mirror's surface itself, behind the mirror, or in front of it; and if it lies in front of the mirror, it may lie between the center of sight and the mirror, at the center of sight itself, or beyond the center of sight. **[2.312-315]** Depending on where they are located, some of the images represent the object clearly, and some do not. **[2.316-323]** PROPOSITION 33: In concave spherical mirrors, when the normal dropped from the object-point intersects the line of reflection, the line-segment from the center of curvature to the object-point will be to the line-segment from the center of curvature to the image-point as the line-segment from the object-point to the endpoint of tangency is to the line-segment from the endpoint of tangency to the image-point. **[2.324-333]** PROPOSITION 34: In concave spherical mirrors, when the object-point and the center of sight lie on the same normal but on opposite sides of the center of curvature, reflection can occur from one to the other from two corresponding points within any given plane of reflection. If the two points lie on different normals, reflection can occur from the arc facing the two points as well as from the arc directly opposite that one, but not from the arcs adjacent to them. **[2.334-337]** PROPOSITION 35: If a diameter is taken on a great circle within a spherical concave mirror, any point on that diameter can be an image-location. **[2.338]** In concave spherical mirrors there can be as many as four and as few as one image, and for each image there is a single corresponding point of reflection.

[2.339-350] PROPOSITION 36: If the object-point and the center of sight lie on the same normal but on opposite sides of the center of curvature, the form of the one can reflect to the other from any point on a circle on the mirror's surface, but if the two points lie at different distances from the center of curvature, there is a limit to how much greater the one distance can be than the other if reflection is to occur. [2.351-373] PROPOSITION 37: If the two points lie on different normals at equal distances from the center of curvature, the form of the one can reflect to the other from one, two, three, or four points on the mirror, depending on where the points are located with respect to the mirror's surface. [3.374-383] PROPOSITION 38: If the two points lie on different normals at different distances from the center of curvature, reflection can occur from only one point on the arc opposite the arc directly facing the two points. [3.384-388] PROPOSITION 39: Given two different normals, and given some point other than the bisection-point on the arc opposite the arc directly facing those normals, that point can serve as a reflection-point for an infinite number of point-pairs on the given normals, provided that those points are not equidistant from the center of curvature. [3.389-391] PROPOSITION 40: Given two points on different normals at unequal distances from the center of curvature, reflection can occur from one to the other from only one point on the arc opposite the arc directly facing those points. [2.392-396] PROPOSITION 41: Given a center of sight and a point of reflection on the arc facing that center of sight, reflection can occur at that point to the center of sight from an infinite number of object-points. [2.397-405] PROPOSITION 42: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if reflection occurs from some point on the arc facing those points, the reflected angle will be greater than or less than the angle adjacent to the angle formed by the normals facing the arc from which reflection occurs. [3.406-409] PROPOSITION 43: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if the form of the one reflects to the other, then the reflected angle cannot be equal to the adjacent angle. [3.410-424] PROPOSITION 44: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if there are two reflections, both reflected angles cannot be less than the adjacent angle. [3.424-438]

PROPOSITION 45: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, it is possible for the form of the one to reflect to the other from two points on the facing arc. [2.439-461] PROPOSITION 46: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, the form of the one can reflect to the other from only two points on the facing arc such that the reflected angle is greater than the adjacent angle. [3.462-464] PROPOSITION 47: Given an object-point and a center of sight lying on different normals at unequal distances from the center of curvature, to find the point or points of reflection at which the reflected angle is greater than the adjacent angle. [2.465-470] PROPOSITION 48: If the object-point and the center of sight lie on different normals at unequal distances from the center of curvature, and if they lie outside the great circle on the mirror formed by the plane of reflection, the form of the one can reflect to the other from only one point on the mirror. [2.471-486] PROPOSITION 49: Depending on whether and how the line connecting the center of sight and the object-point cuts the great circle formed by the plane of reflection on the mirror, there can be as few as one and as many as four reflections. [2.487] For every point of reflection in concave spherical mirrors there will be an image. [2.488-489] It is extremely difficult to see all the images formed in concave spherical mirrors because the head blocks many of them. [2.490] Even though each image is seen separately by each of the two eyes, the object appears single in concave spherical mirrors.

[2.491-519] *Mathematical Analysis of Concave Cylindrical Mirrors . . .* 475

[2.491] Preliminary observations. [2.492-499] PROPOSITION 50: If the plane of reflection cuts an elliptical section on the surface of the cylinder composing the mirror, and if the center of sight and a point of reflection are determined within that section, then, depending on where the object-point is located on some given normal, the image-location can lie behind the mirror, between the mirror and the center of sight, at the center of sight itself, or beyond the center of sight. [2.500-503] PROPOSITION 51: If the object-point and the center of sight lie on the axis of the cylinder from which the mirror is composed, the form of the one will reflect to the other from any point on a circle on the mirror's surface, and the resulting images will lie on a circle outside the surface of the mirror. [2.504-519] PROPOSITION 52: Given an object-point and a center of sight facing a concave cylindrical

mirror, to find the point or points of reflection.

[2.520-547] *Mathematical Analysis of Concave Conical Mirrors* 481

[2.520-523] Preliminary observations. **[2.524-527]** PROPOSITION 53: If the object-point and the center of sight lie on the axis of the cone from which the mirror is composed, the form of the one will reflect to the other from any point on a circle on the mirror's surface, and the resulting images will lie on a circle outside the mirror's surface. **[2.528-547]** PROPOSITION 54: Given an object-point and a center of sight facing a concave conical mirror, to find the point or points of reflection on the mirror's surface.

[BOOK FIVE]

This book is divided into two parts. The first part constitutes the book's prologue; the second [focuses] on images.

[CHAPTER 1]

The first part

[1.1] It is clear from the fourth book that the forms of visible objects are reflected from polished bodies, and sight apprehends them in polished bodies according to reflection. And it has been shown how objects are apprehended through the reflection of [their] forms. Moreover, sight perceives the visible object at a determinate and principal location of reflection when there is no change in the spatial disposition of the visible object with respect to the eye. The form perceived in a polished body is called an image. And in this book we shall explain image-locations in polished bodies, and we shall discuss how to gain a scientific understanding of these locations, as well as how to establish that science by demonstration, and [we shall also discuss] how [those] image-locations are rationally determined.

[CHAPTER 2]

The second part, in which images are discussed

[2.1] The image-location of any point is the point where the line of reflection intersects the normal imagined [to extend] from a point on a visible object to the line tangent to the common section of the surface of the mirror and the plane of reflection, or [to the common section] of the plane that coincides with [the plane of the] mirror and the plane of reflection. We shall demonstrate [this point as follows].¹

[2.2] Take a plane mirror, set it up horizontally, and stand a smooth, straight wooden rod upright upon the mirror. Let the mirror be large enough

that the whole rod can be seen [in it], for, unless the whole rod is visible, error can arise. Let a black spot be marked on the rod. To the eye a rod the same size as this one will appear behind the mirror and directly in line with the actual rod and orthogonal to the mirror, and in the rod that is seen [in the mirror], the black spot marked [on the actual rod] will appear to lie the same distance behind the mirror's surface as [it lies] on the rod above that surface. Furthermore, if the [actual] rod is inclined to the mirror, the rod seen [in the mirror] will appear inclined with the same slant, and the spot marked on the rod seen [in the mirror] will appear to lie the same distance from the mirror's surface as the spot marked [on the actual rod]. Moreover, if some [other] rod is dropped orthogonally to the mirror from the spot marked [on the actual rod], this rod will also appear [behind and] orthogonal to the mirror from the point on the apparent rod, and [it will appear] right in line with the actual rod. The same thing happens for multiple points marked on the rod. The very same thing will happen if the mirror is raised or lowered [by tilting].

[2.3] From this, then, it is evident that the image of a visible point appears on the normal extended from the visible point to the surface of the mirror, and that in this mirror what is perpendicular to the mirror's surface is perpendicular to the common section of the mirror's surface and the [plane of] reflection.

[2.4] The same thing can be shown in the case of a right cone placed upright with its base lying flat upon the plane mirror, for [in that case] another cone will appear [behind the mirror] directly in line with it and sharing the same base, the vertices of these cones lying the same distance from the [surface of the] mirror. And it is obvious that, if a straight line is extended from vertex to vertex, it will be perpendicular to the [shared] base, and thus to the [surface of the] mirror, since that surface and the base [of the cones] coincide, so the [image of] the cone's vertex will appear on the normal dropped from that vertex to the mirror. Likewise, from some point on the cone draw a line to the corresponding point on the cone seen [behind the mirror]. That line will be orthogonal to both the [cone's] base and the mirror's surface, so the image of any point on the cone lies on the perpendicular imagined [to extend] from that point to the mirror's surface.²

[2.5] But whatever point on a body may face a plane mirror, a cone can be imagined with its vertex at that point, and this cone is orthogonal to its base as well as to the surface of the mirror or its continuation. And another cone opposite this one can be imagined sharing the same base and orthogonal to the mirror, and the perpendicular [extended] from vertex to vertex [of the two cones] will be orthogonal to the mirror, so the image of any point facing the mirror lies on the normal [extended] from the point to the

surface of the mirror or to its extension. But it is obvious that in [all] mirrors the perception of forms will be attained only along lines of reflection, so the image of the point seen [in the mirror] lies on a line of reflection, and every such line is straight, so the image of any point [seen in a mirror] lies at the intersection-point of the normal [dropped] from that point to the surface of the mirror and the line of reflection. And it falls on plane mirrors, and the common section of the mirror's surface and the plane of reflection is unique [in coinciding] with the line tangent to the point of reflection, so it is evident that in plane mirrors the appropriate image-location is the point where the normal [dropped] from the visible point to the line tangent to the common section of the mirror's surface and the plane of reflection intersects the line of reflection.³

[2.6] In the case of spherical mirrors that are polished on the outer surface [i.e., convex] what we have claimed will be evident. Find [such] a mirror with a surface large enough that the [entire] form of the thin rod [just used with the plane mirror] can be seen when the rod is stood upright on it. The form of the rod [seen in the mirror] will appear directly in line with the rod [itself], and on the form of the rod [the form of] the point marked on it will appear [to lie] the same [relative] distance behind the mirror's surface as it lies [above the mirror] on the actual rod.⁴ Moreover, if the rod is narrower at one end than at the other, its form will appear conical in this kind of mirror, but this involves a visual illusion that we will explain later.⁵

[2.7] Now, along with this rod form a cone that is orthogonal to a circular base that is rounded off perfectly, and let it also be applied to this mirror. A cone continuous with this one will appear [behind the mirror and] standing on the same base, but [it will appear] smaller than the [actual] cone. That the image should appear conical is clear from the fact that all the lines [extended] from the vertex of the cone seen [in the mirror] to the circle at the base appear equal, and if the cone is moved a bit on the mirror's surface away from where all of it is visible, i.e., so that some of it disappears from view, then, as long as the point of reflection on the mirror is exposed to sight, the image of the cone will be visible.⁶ And if the eye is drawn away from the mirror or approaches it, provided it follows the [original] line of reflection extending between it and the point of reflection, the image of the cone will be visible, but the movement [of the eye] toward or away will be along this line in such a way that the point of reflection is marked. Extend a line from the location of the eye to the mark according to which the movement occurs.

[2.8] Now, since the image of the cone is orthogonal to the cone's base, and since that base forms one of [an infinite number of possible] circles on the sphere, the line [extended] from the vertex of the [actual] cone to the

vertex of its image will be orthogonal to that circle, and it will pass through its center. It will also be orthogonal to the sphere and will pass through the sphere's center, and it will be orthogonal to the plane tangent to the sphere at the point where this line passes through [its surface]. Likewise, it will be orthogonal to the line tangent to the [great] circle on the sphere that passes through that point, and this tangent forms the common section of the plane of reflection and the plane tangent to the sphere at that point, and this line is tangent to the circle on the sphere that forms the common section of the sphere's surface and the plane of reflection. Thus, the line extended from the vertex of the cone to the vertex of its image is perpendicular to the line tangent to the common section of the plane of reflection and the mirror's surface, and that section is a circle.

[2.9] Therefore, the image of the vertex will be seen on this perpendicular, and it is evident that the image of the vertex lies on the line of reflection, so the image of the vertex will be perceived at the intersection of the line of reflection and the normal dropped from the [actual] vertex to the sphere or to the tangent to the circle forming the common section of the sphere's surface and the plane of reflection. Furthermore, from any given point facing this [sort of] mirror a cone can be imagined orthogonal to the mirror's surface or to its extension, and the vertex of this cone is the given point. The line from that point to the image of that point will lie in the plane of reflection as well as on the normal to the mirror's surface or its extension in the way discussed before, for the point seen and its image always lie in the plane of reflection, and so does the line extended from the point seen to its image.

[2.10] In the case of cylindrical mirrors [polished] on the outer surface, the things we claimed about the rod and the cone [in the previous example] are not observed, because in these mirrors what is straight does not appear straight, and there occurs a visual error whose cause we will explain later.⁷ Still, for [any] single point on a visible object [seen in such a mirror] the image-location that has been described can be observed.

[2.11] For instance, having set up the apparatus [described] in the previous book, insert the panel with the [convex] cylindrical mirror attached in it so that the midline of the portion of the mirror [contained by the panel] lies in the plane of the panel. And do not let this panel pass [below] the bronze plaque, but let it stand orthogonally to it so that the [bottom endpoint of] the line along the height of the panel stands on the line that bisects the triangle of the bronze plaque. When the panel is set up on this plaque, fill [the space below] with wax and level the wax so that it lies in the same plane as the [top surface of the] plaque, and this is [done] so that the panel can be placed perfectly perpendicular to the plaque.⁸

[2.12] Then find a pointed ruler, sharpen its edge, and lay this sharpened edge along the midline of the surface of the [cylindrical] ring.⁹ Let it reach along this line [to the mirror], and mark the spot where it touches the [mirror in the] panel. Then lay a needle along this line, and on its end let a small white object be attached, and do not let the needle reach all the way to the [mirror in the] panel.

[2.13] Let the eye[s] be placed in the plane of the ruler, and let one of them be closed. The image of the [small white] object [at the end of the needle] will be seen along the line extended from the point marked [on the mirror] to the point of the needle, and that line is of course perpendicular to the surface of the panel, which is tangent to the cylinder [from which the mirror is formed] along its line of longitude; and that line is perpendicular to the cylinder's line of longitude, which lies in the plane of the panel and forms the common section of the cylinder's surface and the plane of reflection. Also the line of [the cylinder's] longitude and the line perpendicular [to it] lie in the plane of reflection.

[2.14] Now, if the location of the eye is shifted, and if it moves along [the outer edge of the upper] surface of the [cylindrical] ring, then, just as before, the [small white] object [at the end of the needle], the image of the object, and the [point of the] needle will appear [to lie] on the same line. Moreover, that line is perpendicular to the midline along the length of the cylinder, and this perpendicular lies in the plane of reflection, for the [top] surface of the [cylindrical] ring intersects the cylinder along a circle that is parallel to the base of the cylinder, and the eye lies in the plane [of this circle]. And later on we shall demonstrate that, when the eye and the visible object lie in a plane parallel to the base of the cylinder, that plane constitutes the plane of reflection.¹⁰ In this situation, however, the common section of the cylinder's surface and the plane of reflection is a circle, and the normal upon which the image and the [small white] object are seen falls orthogonally to a line tangent to the circle.¹¹

[2.15] When all this is done, remove the needle and place the sharp-edged ruler along the midline [of the cylindrical ring's upper surface] so that it reaches the midline along the longitude of the panel, and attach the sharp-edged ruler firmly to the [cylindrical] ring's [upper] surface with wax. Then remove the panel with the mirror in it and take a pointed ruler and lay its sharpened edge upon the midline along the length of the panel, and along the sharpened edge draw a line upon the mirror with ink. Then take a moderate-sized wax triangle, one of whose sides is equal to the height of the panel containing the mirror, let it be moderately thick, and let the surface of this triangle be as flat as possible. Then attach the wax triangle firmly to the panel containing the cylinder below the panel's base, and place the

side of the triangle that is equal to the height of the panel upon the side of the panel's base. When that is done, the length of this triangle at the base of the [panel containing the] cylinder will be equal to the [height of the] panel, and in order to make the triangle's surface as flat as the surface of the panel, place the triangle between a panel and a flat surface and compress it until it is quite flat. Place a sharp-edged ruler on the surface of this triangle, and cut off the bottom edge of this triangle along the edge of the ruler, and this edge will form a straight line. This line will form the base for the panel containing the mirror.

[2.16] Then place the panel on the surface of the [bronze] plaque in the instrument, and place the edge of its base, which lies along the length formed by the side of the wax triangle, on the line of longitude of the bronze [plaque], as was done before. The plane of the panel containing the mirror will be perpendicular to the bronze plaque, and this plane cuts the bronze plaque along the line of longitude of the bronze plaque, and this plane is tangent to the surface of the mirror along the line on the mirror's surface. And this plane is the surface of the panel containing the mirror, and the angle [along the edge] of the pointed ruler will be attached along the midline of the [top] surface of the ring, and the mirror's surface will be tilted downward to that surface on the side of the triangle's vertex, since one side of the panel has been raised according to the breadth of the triangle, while the other side [of the triangle] beyond the vertex of the triangle is the plane of the bronze [plaque], and the line that lies along the midline of the mirror will be inclined.

[2.17] And when the side of the wax triangle is [positioned] along the line of longitude of the bronze [plaque], the panel that contains the mirror will move along the line of longitude of the bronze plaque, and the [base] side of the [wax] triangle will move with it, if it is [positioned] on the [bronze plaque's] line of longitude. And let it move back and forth until the point of the sharp-edged ruler meets some point on the [mid]line on the surface of the mirror and the mirror fits snugly against the sharp-edged ruler, and erase the line drawn in ink on the mirror. Mark the point on the surface of the mirror that is directly under the end of the sharp-edged ruler. And remove the sharp-edged ruler, and apply the needle, and let the needle lie along the midline of the [top] surface of the ring, and let it be attached firmly with wax. And the imaginary line from the point of the needle to the point marked on the surface of the mirror will be perpendicular to the plane of the panel which is tangent to the mirror's surface at the marked point, and it will lie perpendicular to any line extended through that point in the plane tangent to the mirror. Thus, it will be perpendicular to the straight line tangent to the common section of the top surface of the ring and the [mirror's] surface.

[2.18] Now, place the eye in the plane of the ring at its apex [on the midline along its top surface], and it will look into the mirror until it perceives the form of the small [white] object on the needle, and then it will perceive that object, the point marked on the mirror, and the image of that object [all in a line]. And the line passing through the small [white] object and the point marked on the [mirror's] surface is perpendicular to the plane tangent to the mirror's surface at the point marked on it. And this surface on the ring is among the planes of reflection, and the small [white] object and the centers of sight lie in that plane, and the point of reflection is in that plane, and we will prove this later on. And in this case the image of the small [white] object will lie upon a straight line extended from the small object [itself] directly to the plane tangent to the mirror's surface, and since this line is perpendicular to the straight line tangent to the common section of the surface of the mirror and the plane of reflection, which is the [top] surface of the ring, and since the plane of reflection is among the oblique planes that cut the cylinder between the lines of longitude on the cylinder and the circles parallel to its bases, then, because the panel and the mirror in it are at a slant, the common section of this plane [of reflection] and the surface of the mirror is among the cylindrical sections [i.e., an ellipse].¹² And we will explain the image-location in this same way if the orientation of the panel containing the mirror is changed and slanted on its surface at a smaller or greater obliquity.

[2.19] It is therefore evident from these things that the image is perceived where the normal extended from the visible point to the surface of the mirror intersects the line of reflection, and this is the predicted location. If lines are drawn from the visible point to the mirror's surface, the one that is perpendicular is shorter than any of the others, for any of the others first intersects the common section of the plane tangent to the mirror's surface to which the perpendicular falls and the plane of reflection that reaches the mirror, and any line extending in this plane from the visible point to this common section is longer than the perpendicular, since it subtends a greater angle, so what was proposed [is demonstrated].¹³

[2.20] The same procedure can be applied in the case of the conical mirror [polished] on the outer surface, and the same thing will be shown whether the images of objects are seen in [planes formed by] conic sections, or whether they are seen in [planes formed by] cuts made along lines of longitude.¹⁴

[2.21] In concave spherical mirrors some images are perceived behind the mirror, some on the mirror's surface, and some in front of the mirror, and some of these images are perceived according to reality, whereas some are perceived in ways that do not accord with reality.¹⁵

[2.22] That all of those images seen according to reality appear at the point where the normal [dropped from the object] intersects the line of re-

flection will be demonstrated as follows. Fashion a cone that is orthogonal to its base [i.e., a right cone], and let the diameter of the base be smaller than the radius of the sphere [from which the concave mirror is formed], and let the line along the length of the cone be longer than that radius. Now, [from the vertex] toward the base, cut off a length [along the cone's line of longitude] equal to that radius, inscribe a circular section [parallel to the cone's base at that point] and cut the cone according to this circle. Then, in the middle of the mirror draw a circle the same size as the base of the cone that is left [after the cut], fit the cone to this circle, and attach it firmly with wax.

[2.23] Place the center of sight where the image of the cone can be perceived, and direct light upon it so that the perception may be clearer.¹⁶ You will not see a cone conjoined with this one, but you will perceive [the image of] this [cone] extending behind the mirror, so a certain cone directly in line [with it] will appear with its base behind the mirror along with the wax section of its cone.¹⁷ And if a line of longitude is marked with ink on this cone, this line will appear to extend along the surface of the cone that appears [in the mirror], and since the vertex of the cone lies at the center of the sphere, the line extending from the vertex along the length of the cone will be perpendicular to the tangent to any circle on the sphere that passes through the endpoint of the line.

[2.24] Hence, any line of longitude on the cone that appears [in the mirror] is perpendicular to the line tangent to the common section of the plane of reflection and the surface of the sphere, that line being the common section, and it is a [great] circle. And any point on the cone [lying] on this perpendicular is seen, and every [such] perpendicular lies in the plane of reflection, because the visible point and its image lie on the normal as well as in this plane of reflection. But every image is perceived along the line of reflection, so the image of any point on the cone will lie at the point of intersection of the normal and the line of reflection.

[2.25] Furthermore, points whose images are perceived in front of the mirror—i.e., ones that lie between the center of sight and the mirror—[are perceived this way] since the line extended from any of them to the center of the mirror will cut the side of the line lying between the visible object and the mirror. In order to observe this [fact], remove the cone from the center of the mirror and replace it at the side. The vertex will be the center of the mirror, and let the distance between the center of sight and the mirror be greater than the sphere's radius. Then take a thin white rod, and stand it on the mirror so that the center of the mirror lies directly between the top of the rod and the center of sight, and direct your line of sight to the point on the mirror from which the line extended to the vertex of the cone lies between the top of the rod and the center of sight. Then, look into the mirror

until neither the top of the rod nor the rod itself appears, and the form of the top of the rod will appear above the mirror and nearer to the eye than the vertex of the cone. And the vertex of the cone, the top of the rod, and the image of the top of the rod will lie on the same straight line, and this line is perpendicular to the line tangent to the common section of the mirror's surface and the plane of reflection. For the plane of reflection passes through the center [of sight] and the visible point, and the line passing through these two points lies in the plane of reflection, and the common section is a circle. And this line will form a diameter in this circle, because the center of that circle is the center of the sphere, so this line will be perpendicular to the line tangent to the circle at the endpoint of this line, and this line passes through the visible point as well as through its image. And so, any point [whose image is seen] in front of the mirror is perceived on the same line as the center [of the sphere] and the image, and any point is seen on the line of reflection, so [it is seen] at the point of intersection of the normal and the line of reflection.

[2.26] And the things that are perceived in these mirrors as they exist in reality are those whose images appear behind the mirror or in front of the mirror, and beyond these [types of images] there are none that sight may perceive as they exist in reality in this [sort of] mirror, for these latter types of objects do not permit their true images to appear. Images that appear on the surface of this [type of] mirror belong to the last category, and we will explain this when the discussion turns to visual errors. Therefore, any point that is perceived as it actually exists in this [sort of] mirror appears at the intersection of the normal and the line of reflection, and this normal passes from the visible point to the center of the sphere and falls orthogonally on the tangent to the common section [of the mirror and the plane of reflection].¹⁸

[2.27] In the case of concave cylindrical mirrors the image varies, for its location will sometimes be on the surface of the mirror, sometimes behind it, and sometimes in front of it. And in all these cases [the image] is sometimes perceived as it actually exists, sometimes not.

[2.28] If you want to observe the image-location in these cases, do what you did [before] in the case of cylindrical mirrors [polished on the] outer surface. Insert the panel with the concave cylindrical mirror on it into the ring, as it was inserted earlier, and likewise [position] the needle and the small [white] object at the end of the needle. Place your eye directly in front of the center of the circle [passing through the middle of the mirror] and at the middle of the [top] surface of the ring, raise the eye a little bit above the ring's [top] surface, and let it look until it sees the image of the object and perceives the form of the object, the object itself, and the point marked on

the mirror along the same line that is perpendicular to the mirror's surface—and [determine] this according to sense-induction.¹⁹ The image will lie behind the mirror, and the reflection will occur from a point on the line [of longitude] that lies in the middle of the mirror.

[2.29] Then place the center of sight in the plane of the [top surface of the] ring, but outside the midline, until it sees the image of the small [white] object [on the needle]. You will see it in front of the mirror, and you will see the object, its image, and the point marked on the mirror along one straight line perpendicular to the straight line tangent to the circle that is parallel to the base of the mirror and [lying] upon the point marked on the mirror's surface. And the plane of this circle forms the plane of reflection in this case, and it is the top surface of the ring, and the point of reflection is a point on that circle.²⁰

[2.30] Afterwards, with your other hand place another needle with a small object attached to its end, and lay it on the [top] surface [of the ring] and the axis in such a way that the object and the point marked [on the mirror] lie on the same line according to sense-induction. Let the eye lie in the plane of the [top surface of the] ring between its endpoint and the middle [of the ring's top surface]. It will see the image of the object, and it will see this image as well as its [generating] object along with the point marked on the surface of the mirror on the same straight line.²¹

[2.31] If, however, the straight line [of longitude in the middle of the panel] is slanted according to the small triangle we fashioned—and the center of sight should be on the midline of the ring—you will see the image in front of the mirror, but on the same straight line as the body and the point marked [on the mirror's surface]. And this reflection will occur from one of the cylindric sections [i.e., ellipses], because the mirror is inclined, and we know that an image is perceived only along a line of reflection. Therefore, it is clear that the image-location lies where the normal intersects the aforementioned line of reflection, since the [image] is perceived properly, and even though the image may not be perceived clearly, these images will still belong to the category of those that are properly seen.

[2.32] In the same way you can observe [the image] in a concave conical mirror at the intersection of the normal with the lines of reflection. It is therefore evident that in all mirrors images are perceived at the aforementioned location, which is likewise referred to as the "image."

[2.33] We will now explain why visible objects are perceived through reflection where the image is located and why the image lies on the normal [dropped] from the visible object to the surface of the mirror. When the visual faculty grasps a form by reflection, it grasps it immediately without

an accurate determination, and it grasps its distance through estimation. It may perceive this distance accurately through a close and careful scrutiny, or it may not. And we have explained this [process] in the second book, where it was claimed that the visual faculty grasps distance through a deduction based on the size of the object and the particular angle under which the size is perceived. The perception of a visible object whose location is unknown clearly occurs in this way. Also, objects [whose locations are] known are perceived in this way, for they are compared to known objects and known sizes or distances.²² When sight perceives some object through reflection, it perceives the distance of the image solely by estimation; then, after a close scrutiny, it grasps the distance and verifies [it] deductively on the basis of the size of the visible object and the angle of the cone according to which the form is reflected to the eye.

[2.34] Thus, if the visible object is among things familiar [to the perceiver], the visual faculty grasps its distance according to an already-known distance [occupied by such a body] subtending an angle equal to this one and [lying at] a distance similar to this one. Likewise, when the visible object is unknown, its magnitude will be compared to another magnitude belonging to familiar objects, and the distance of this image is grasped according to a deduction based on the size of the angle that the image subtends at the center of sight at the time of reflection. Moreover, the place where the form of the visible object is perceived through reflection [is such that] the form reaching the eye directly from it at a [given visual] angle will arrive according to the very same cone according to which the form is reflected to the eye, and the same cone will encompass the entire form as it does at the image-location. Hence, when it grasps the visible object through reflection, the visual faculty grasps it where the image lies, for, because the form perceived through reflection at the image-location is identical to the form as perceived directly, it is contained by that [same] cone, and this is why it is perceived at the image-location.²³

[2.35] We will now explain why the image will be perceived on the normal. We know that a point that is perceptible to sight exists not intellectually but sensibly, and its form is sensible. I say therefore that in the case of plane mirrors, since the image does not appear on the surface of the mirror but behind it, it is more appropriate and reasonable for it to appear upon rather than outside the normal. For, since it has been defined according to a spot on the normal, the image's distance from the point of reflection on the mirror—which is a segment of the line of reflection extended from the location of the image to the point of reflection—is equal to the distance of the point that is seen from the point of reflection. Since the surface of the mirror is orthogonal to the normal, then the line extended [on that surface] from

the point of reflection to the normal forms a side common to the two triangles, and it will bisect the normal. Hence, two sides of one of the triangles will be equal to two sides of the other, and an angle [of one will be equal] to an angle [of the other], for both are right, so the base [of the one will be equal] to the base of the other.²⁴

[2.36] Thus, if the image appears on the normal, it will lie the same distance from the point [of reflection] and the eye as the object from which it originates, and the image will occupy an equivalent location with respect to the point of reflection as the point that is seen does with respect to the same point [of reflection], and the location [of both the image-point and object-point] will be equivalent with respect to the center of sight, so in this situation both the point seen and its image will appear according to reality. If, however, the image lay beyond the normal, then, since it must lie on the line of reflection, it will lie either beyond or in front of the normal with respect to the eye. If the image lies beyond, then it will lie farther from the point of reflection and from the eye than the point seen, so it will subtend a smaller angle in the eye than the point [itself]. Moreover, it occupies a smaller area on the eye, so, when it should be equal, it will appear smaller than it[s generating object]. On the other hand, if it lies in front of the normal, it will appear larger [than its generating object-point] because it lies nearer.²⁵

[2.37] In the case of a convex spherical mirror, the image appears on the normal, for the image that appears is either of the center of the eye or of some other point.²⁶ If the image is of the center [of the eye], then I say that it is more appropriate for the image of the center of the eye to appear upon the normal dropped from the [center of the] eye to the center of the sphere than [to appear] anywhere else, for if the form propagates directly along this normal to the center of the sphere, then it will always maintain a uniform disposition with respect to the eye, and so the form that faces any point on the sphere and moves orthogonally toward the center will maintain a uniform disposition with respect to the eye. And the disposition of the form will be the same on one normal as it is on another, for the center of the sphere has the same situation with respect to every point on the sphere, and all normals of this sort are identically situated.

[2.38] On the other hand, if the image propagates to some point on the sphere outside the normal, its disposition with respect to the eye will change, for it will have a different situation outside the normal than it does on the normal itself, and the normal moves outside the mirror rather than inside it. But if it appears outside the normal, it will not maintain a [uniform] disposition, yet it was [agreed earlier that it is] more fitting for the image to maintain [a uniform] disposition than for it to change [its situation] if the visual faculty is to perceive the visible object with certainty. Accordingly, the im-

age of the [eye's] center appears on the normal, yet we cannot [on that account] determine a specific point for that image on the normal, because there is nothing to be found in any point on the normal to give it precedence over any other point in determining where the image should appear on it. But we know that at whatever point on this normal the image [of the eye's center] appears, it invariably appears continuous with the [whole] eye appearing [in the mirror], and it always maintains [the same] location and situation with respect to the entire form [of the eye] that is seen.

[2.39] The image of any point on the eye other than the center reaches [the mirror] obliquely, so it does not maintain a uniform disposition with respect to the object-point that is seen [i.e., the center of sight], and the normal dropped from the point seen to the mirror[']s surface] falls to the center of the sphere, and it is upon that normal that the image maintains a uniform situation. There is thus no point where the perceived image will maintain a uniform disposition except upon that normal, and since it must be perceived along the line of reflection, it will be perceived at the intersection of this line with that normal.²⁷ We have thus explained the reason for this phenomenon, but the [essential] character of natural objects is related to the [essential] character of their [underlying] principles, and the [underlying] principles of natural objects are hidden.²⁸

[2.40] The same way of proving this will obtain in the case of a concave spherical mirror, as well as in the case of a concave or convex conical [mirror]. In general, for any kind of mirror, image-location will be on the normal, for there is nowhere outside the normal where the form will maintain a uniform and equivalent disposition.

[2.41] With these points clarified, it remains for us to rationally define the image-location for every kind of mirror.

[2.42] We say that, for each and every point that is perceived by sight in a plane mirror, when it lies outside the normal that falls from the center of sight to the surface of the plane mirror, the line along which the form of that point is reflected to the center of sight will intersect the normal dropped to the mirror's surface from that point. Moreover, this intersection-point, which constitutes the image-location [of the object-point], lies behind the mirror[']s surface], and this [image] will lie the same distance from the mirror's surface as the [object]-point viewed [in the mirror]. And the visual faculty perceives the image of the point viewed [in the mirror] only in that location, and only one image of any point perceived by sight in such a mirror will be seen.

[2.43] Moreover, for any point that sight perceives in a convex [spherical] mirror, when its form radiates outside the normal dropped from the

center of sight to the center of the mirror, the line along which the image is reflected to the eye will intersect the line extended from that point to the center of the mirror, that line being the normal dropped from that point, and it is orthogonal to the line tangent to the common section of the plane of reflection and the surface of the mirror. Moreover, where the point of intersection (which constitutes the image-location) lies with respect to the mirror's surface will depend upon where the center of sight lies with respect to the mirror's surface. The intersection-point may lie beyond the mirror, on the mirror's surface, or behind the mirror[*'s surface*]. But the visual faculty perceives all [these] images behind the mirror, even though their locations may vary, and it does not perceive the location of any image on the mirror's surface through deduction. Also, no matter what point is perceived by sight in this sort of mirror, it yields only one image.²⁹

[2.44] In the case of a convex cylindrical mirror (and the same holds for a convex conical mirror), if any given point that is perceived by sight lies outside the normal dropped from the center of sight orthogonal to the plane tangent to the mirror's surface, the line along which the form is reflected to the center of sight intersects the normal dropped directly from that point to the line tangent to the common section of the plane of reflection and the surface of the mirror. And some of the images in these mirrors lie beyond the mirror's surface, some on the surface itself, and some inside it.³⁰ But the visual faculty grasps all of the images in these mirrors behind the mirror's surface, and whatever point the visual faculty may perceive in these mirrors produces only one image.

[2.45] In the case of a concave spherical mirror, some of the lines along which the forms of visible points are reflected intersect the normals extended straight from those points to the lines tangent to the common sections of the surface of the mirror and the plane of reflection, and some are parallel to these normals. For those that intersect the normals, the point of intersection, which constitutes the image-location, sometimes lies behind the mirror outside the point of intersection—and this perception is unclear.³¹ Moreover, of the visible objects that sight grasps in this [sort of mirror], some yield one image only, some two, some three, and some four, but there can be no more [than four].

[2.46] In a concave conical or a concave cylindrical mirror, some of the lines along which forms are reflected intersect the normals extended straight from the visible points to the lines tangent to the common sections [of the mirror's surface and the plane of reflection], and some are parallel to the normals. In the case of those that intersect the normals, the intersection for some of the points lies behind the mirror, and the intersection for others lies

in front of the mirror. Some of those that lie in front of the mirror will lie between the mirror and the eye, some at the center of sight [itself], and some beyond the center of sight. The perception by sight of visible objects in this [sort of mirror] sometimes occurs at the [proper] image-location, which is the point of intersection, and sometimes outside the point of intersection. And of those objects that are perceived [in this sort of mirror], one yields a single image only, another two images, another three, and another four, but there is no way that a thing can yield more than four. And we will demonstrate all these points theoremtically.

[Plane Mirrors]

[2.47] **[PROPOSITION 1]** Let A [figure 5.2.1, p. 220] be the object-point, B the center of sight, and DGH a plane mirror. Let G be the point of reflection and DGH the common section of the plane of reflection and the mirror's surface. From point G draw EG perpendicular to that common section. From point A draw AH perpendicular to the mirror's surface, and continue it behind the mirror. Let AG be the line along which the form [of point A] reaches the mirror and BG the line along which it is reflected to the center of sight. Accordingly, BG, EG, and AG lie in the plane of reflection, and since [normal] EG is parallel to [normal] AH, while BG is oblique to EG, BG will intersect AH. Let it then intersect at point Z. I say that ZH = HA.

[2.48] Since angle BGD = angle AGH,³² [since angle AGH = angle HGZ, which = vertical angle BGD], since [right] angle AHG = [right] angle ZHG, and since side HG is common, then [by Euclid, I.26] triangle [AHG] = triangle [ZHG]. Hence, ZH = AH.

[2.49] Now, if we wish to determine where the point of reflection lies according to normal [AHZ], let a segment along the normal be cut off below the mirror equal to the segment on it [from object-point A] to the mirror, i.e., so that ZH = AH. Then extend line BGZ from the center of sight to point Z. I say that G is the point of reflection.

[2.50] Since AH and HG = HG and HZ [leaving AH = HZ and HG common], and since [right] angle [AHG] = [right] angle [ZHG], then [by Euclid, I.4] triangle [AHG] = triangle [ZHG]. Therefore, angle ZGH = angle HGA. But [vertical angle] ZGH = [vertical] angle DGB. It follows, therefore, that angle BGE [in right angle EGD] = angle EGA [in right angle EGH], and so G is the point of reflection. And what was proposed [has] thus [been demonstrated].

[2.51] **[PROPOSITION 2]** Furthermore, let A [figure 5.2.2, p. 220] be the center of sight, let AG be the normal to the plane mirror, and let D [be the point] on the eye's surface [where it] intersects this normal. I say that

there is no point other than D on this normal that may reflect from this mirror to the eye.

[2.52] For if [some such] point may be taken on this normal beyond the eye, let it be H. Its form will not reach the mirror along the normal because of the interference of the opaque body [of the eyeball], and so its form is not reflected along the normal.

[2.53] But if it is claimed that [its form] can be reflected from some other point on the mirror, let that [point] be B. Its form will propagate to point B along line HB, and it is reflected along line BA. Let angle HBA be bisected by line TB [so that HBT is the angle of incidence and TBA the angle of reflection]. Therefore, TB will be normal to the mirror's surface. But TG is normal to that same [surface], so two perpendiculars will have been dropped from the same point [T] to the mirror's surface, which is impossible.

[2.54] The demonstration that the form of point D cannot be reflected from any point other than G on the mirror will be the same, so that form is reflected only along the normal. If, however, it is claimed that the form of any point selected between G and D on this normal is conveyed to the center of sight by reflection, the disproof follows from the fact that the body [of the selected point] will be either opaque or transparent.

[2.55] If it is opaque, then the point's form will proceed to the mirror along the normal and will return along the same line to the center of sight, but because of the [point's] opacity that form cannot pass through to reach the center of sight.

[2.56] On the other hand, if that point is transparent, its form will mingle with it and fuse with it upon its return from the mirror along the normal, and it will not be reflected to the center of sight.

[2.57] Moreover, by the preceding method, it can be proven that the form of any point selected on this normal between G and D cannot be reflected to the center of sight from any point on the mirror other [than G]. By the same token, the form of [any] point selected between A and D is reflected to the center of sight neither along the normal nor along any other line, for the points lying between the center of the eye and its surface are absolutely transparent, so their form is neither conveyed back nor reflected in such a way as to be sensed. And since no point other than D selected on the surface of the eye faces the mirror at a right angle, any such point will be seen upon the normal dropped from it to the mirror, and its image [will lie] the same distance behind the mirror's surface as the point itself [lies above it]. And since D appears continuous with the other points on the surface of the eye, and since its image appears continuous with the other images [of those points], the image of D will appear [to lie] the same distance from the mirror's surface as D lies from it.³³

[2.58] Hence, it is evident that the image of any point seen in the mirror will appear on the normal, and the distance of the image from the mirror's surface and [the distance] of the visible body [from the mirror's surface] is the same.

[2.59] **[PROPOSITION 3]** Furthermore, the form of a point viewed in a plane mirror is reflected to the same center of sight from only one point. Let A [figure 5.2.3, p. 220] be the center of sight, B the point viewed, and ZH the mirror. If, then, it is claimed that the form of B is reflected to the center of sight from two points on the mirror, let one of them be point D, the other E. Draw line AB from the point viewed to the center of sight; this line will either be perpendicular to the mirror or not.

[2.60] If it is not perpendicular, we know that the line lies in the plane of reflection [which is] orthogonal to the surface of the mirror, and it will lie in only one such plane. For if [it lies] in two, it will be common to two orthogonal planes, and if a point is chosen on it and a line is extended from it in either plane to the common section of that plane and the surface of the mirror, this line will certainly be orthogonal to the mirror. Likewise, from that [same] point extend a line in the other plane to the common section of that plane and the surface of the mirror, and this line will be orthogonal to the mirror, so from the same point two perpendiculars will have been dropped [to the same plane].³⁴

[2.61] Thus, since BA lies in only one orthogonal plane, and since the three points A, B, and E lie in the same orthogonal plane, AE and EB will also lie in the orthogonal plane that contains AB, and so will EB and DB. Therefore, EA and EB [will lie] in the same plane as DA and DB. But angle AEH = angle DEB [by supposition], and angle HEA > angle ADE, because it is an exterior [angle of triangle AED], so [angle] BED > [angle] ADE. But [angle] BDZ = [angle] ADE [by supposition], while [exterior angle] BDZ [of triangle BED] > [interior angle] BED, so [angle] ADE [which = angle BDZ, by supposition] > [angle] BED, but it was [just] claimed that it is smaller. It follows, therefore, that reflection may occur from only one point.

[2.62] On the other hand, if AB is perpendicular to the mirror, then it has already been claimed [in proposition 2] that there is only one point on the line dropped orthogonally from the center of sight to the mirror whose form is reflected from the mirror to the center of sight. And it has already been demonstrated that the image of that point reflects from only one point [on the mirror], so what was proposed [has been demonstrated].

[2.63] **[PROPOSITION 4]** Furthermore, when any point is viewed by both eyes, one and the same image appears to both, and it does so in the aforementioned [image]-location. Now, it is obvious that the form of the

point does not reflect to both eyes from the same point on the mirror. For if the line of reflection proceeding to one eye were to form the same angle with the normal erected to the mirror's surface [at the point of reflection] as the line along which the form reaches the mirror, then another line could not be chosen in the same plane to form with the normal an angle equal to this one.³⁵ Hence, from this point no [form incident along the same] line will be reflected to the other eye. Reflection must therefore occur from different points on the mirror.

[2.64] Let those points be T and Z [figure 5.2.4, p. 221]. Let QE be the plane mirror, A the point viewed, B and G the two centers of sight, and AD the normal [dropped from object-point A to the mirror]. It is evident, then, that BT, AT, [segment DT of] ET, and AD lie in the same plane orthogonal to the surface of the mirror. Likewise, AD, AZ, and GZ lie in the same orthogonal plane, and DT is the common section of plane ADTB [and the mirror's surface], while DZ is the common section of plane ADZG [and the mirror's surface]. If BT and GZ lie in the same orthogonal plane, TDZ will form a single line, and normal AD will lie between the two aforementioned perpendiculars [dropped] to the mirror's surface from the two centers of sight [i.e., the perpendiculars dropped from B and G to the plane of the mirror containing line QDE], or it will lie outside them [figure 5.2.4a, p. 221].

[2.65] Whichever the case, line BT of reflection will cut from normal AD a segment behind the mirror equal to segment AD [above it, as demonstrated in proposition 1 above]. Likewise, GZ will cut from the same normal a segment behind the mirror equal to that segment [AD above the mirror]. Those two lines of reflection will therefore intersect the normal at the same point [H] behind the mirror. Hence, the image of point A will be perceived by both eyes at the same point on the normal, so there will only be one image, and it will be the same [for both eyes], and it will lie at the same place as it would if it were viewed by only one eye.

[2.66] If, however, points T and Z do not lie in the same plane orthogonal to the mirror [figure 5.2.4b, p. 221], precisely the same proof will apply, since each line of reflection cuts a segment on the normal [below the mirror] equal to the segment above it, and the intersection of the lines of reflection with the normal will be at the same point, so what has been proposed [is demonstrated].

[2.67] On the other hand, if point A lies on the normal dropped to the surface of the mirror from only one center of sight, it is perceived by that same center of sight behind the mirror at a point on the normal that lies as far from the mirror's surface [below it] as A lies from it, so the form of A appears continuous with the forms of other points that appear in proximate

locations. Also, the image of A is perceived by the other eye at the same point on the normal, and so only one image of point A is seen by both eyes, and [it appears] at the same point on the same normal, which is what was proposed.³⁶

[Convex Spherical Mirrors]

[2.68] **[PROPOSITION 5]** What we claim will be clear in the case of spherical convex mirrors. Let A [figure 5.2.5, p. 222] be the point seen, B the center of sight, and G the point of reflection. It is clear that BG and AG lie in a plane [of reflection] orthogonal to the plane tangent to the sphere at point G. Let ZGQ be the [great] circle that forms the common section of the plane of reflection and the surface of the sphere [from which the mirror is formed]. Let PGE be the line tangent to this circle at the point of reflection. Let HG be the normal to this line [at the point of reflection]. It is evident that HG should reach the center of the circle. But if not, then, since the line extended from the center of the sphere to point G is also perpendicular to line PGE, two lines perpendicular to one [and the same] line will have been drawn from the same point on the same side [of that line].³⁷

[2.69] Now, let N be the center of the sphere, and let a line, i.e., AN, be extended from the point viewed to the center of the sphere, that line being normal to the plane tangent to the sphere at the point on the sphere through which it passes. And since it is manifest that [line of reflection] BG intersects the sphere, because it lies between HG and GP, which form a right angle, it will intersect line AN. And since normal HG lies in the plane of reflection, the center of the sphere will lie in the same plane, and so [normal] AN [will lie] in the same plane as [normal] HG.

[2.70] Accordingly, let D be the intersection of [line of reflection] BG with [normal] AN. It is clear that D will be the image-location, and this analysis must be understood [to obtain] when the line extended from the point seen to the center of sight is not perpendicular to the mirror.

[2.71] **[PROPOSITION 6]** Now, let line PGE [figure 5.2.6, p. 222] intersect line AN. Let the point of intersection be E, and this point is designated as the endpoint of tangency. I say that in this case the line extending from the center of the sphere to the image-location is longer than the line extending from the image-location to the point of reflection: that is, $DN > DG$.

[2.72] For angle [of reflection] BGH = angle [of incidence] HGA [by previous supposition], but angle BGH = [vertical] angle NGD. Therefore, angle [of incidence] HGA = that same [angle NGD], and EG is perpendicular to HGN, so angle AGE = angle EGD [because angle AGE = angle BGP = vertical angle EGD]. Therefore, $AG:DG = AE:ED$ [by Euclid, VI.3].

[2.73] Let a line [AH] be drawn from point A parallel to DG, and let it intersect line HN at point H. Accordingly, angle NGD = [alternate] angle GHA. But angle NGD = angle AGH [by previous conclusions]. Therefore, angle GHA = that same angle [AGH], so the two sides AG and HA [of triangle AGH] are equal. Therefore, AH:DG = AG:DG. But AH:DG = AN:DN [by Euclid, VI.4, because triangles AHN and DGN are similar], so AN:DN = AG:DG. Therefore, AN:AG = DN:DG [by Euclid, V.16]. But AN > AG, because it subtends more than a right angle in triangle ANG [by Euclid, I.19 and 32]. Accordingly, DN > DG, which is what was proposed.

[2.74] **[PROPOSITION 7]** I say, further, that the line dropped orthogonally to the sphere from endpoint of tangency E, i.e., the segment [EF] of line EN [between E and the surface of the sphere], is shorter than the radius [of the sphere].

[2.75] Let F [figure 5.2.7, p. 223] be the point where [normal] AN [passing through endpoint of tangency E] intersects the sphere's surface. I say that EF < NF.

[2.76] For, as has [already] been claimed [in the previous proposition], AG:DG = AE:ED, but AN:DN = AG:GD [by previous conclusions]. Therefore, AN:DN = AE:ED. Therefore, AN:AE = DN:DE [by Euclid, V.16]. But AN > AE, so DN > DE, and so DN > EF, from which it follows that NF > EF, which is what was proposed.³⁸

[2.77] **[PROPOSITION 8]** Now, let G [figure 5.2.8, p. 223] be the center of sight, D the center of the sphere, and DZG the normal [dropped] from the center of sight to the sphere. I say that the form of no point other than the point on the surface of the eye [through which normal GD passes] is reflected along this normal.

[2.78] For the forms of the points selected beyond the center of the eye are not reflected for the reason given above [in proposition 2]. And the same holds for the points lying between the surface of the eye and the mirror. I say, as well, that no point on this normal is reflected from any other point on the mirror.

[2.79] For if it is claimed that [reflection does occur] from another point, let that point be A. Line GA will be the line of reflection, and from that point [on the normal, which is assumed to reflect its form to G, i.e., X] we will imagine a line to A, that line [AX] being the one along which the form proceeds [to the mirror]. These two lines form an angle at A, and diameter DA will necessarily divide that angle, since it is normal to point A, and the normal bisects the angle produced by the form's line of incidence and its line of reflection. And so diameter DA will intersect normal GD between

the selected point and G [i.e., at point E]. And so the two straight lines [GD and DE, which consists of DA and its rectilinear continuation AE] will intersect at two points [D and E] and will form a plane.

[2.80] It therefore follows [from the impossibility of such a double intersection] that the form of only the one point that lies on the surface of the eye may be reflected from the mirror along the normal, and it must appear at its original image-location according to its continuity with other [surrounding] points.³⁹

[2.81] **[PROPOSITION 9]** Furthermore, let GA and GB [figure 5.2.9, p. 224] be lines drawn tangent to the sphere from the center of sight, and mark off the [great] circle upon which the plane formed by these lines cuts the sphere. AB will be the visible portion of this circle. I say, therefore, that some of the image-locations perceived according to reflection from this portion lie inside the mirror, some on the mirror's surface, and some outside the mirror. Each one of these cases must be accounted for.

[CASE 1]

[2.82] From point G [figure 5.2.9] draw a line that cuts the circle, and let the segment of it that forms the chord on the [intersected] arc of the circle be equal to the circle's radius. Let that [cutting] line be GHK, and let HK be the [segment on it forming the] chord that is equal to the radius. Then, from point H draw normal DHM. I say that the [image-]location of a form reflected from point H will lie inside the sphere.

[2.83] From point H draw a line OH that forms with MH an angle [OHM] equal to angle MHG. Points [incident to the mirror] on this line, and on no other, will be reflected from point H to the center of sight [G]. Accordingly, choose some point O on it, and draw line OD [i.e., the normal dropped from the object-point] from that point to the center of the sphere. OD will be perpendicular to the plane tangent to the sphere at the point where OD passes through it. But, by construction, angle OHM = angle MHG, so, by the same token, it is equal to [angle MHG's] alternate angle KHD. But [angle] KHD = [angle] KDH, because they are subtended by equal sides [of equilateral triangle KDH, i.e., radii DH and DK].

[2.84] Accordingly, angle OHM = angle KDM, so lines KD and OH are parallel. Therefore, if they are extended indefinitely, they will never intersect.⁴⁰ But line OD will intersect the line [HK] joining KD and OH, and so, no matter what point is chosen on line OH, the line extended from that point to point D [i.e., the normal] will intersect the line of reflection inside the sphere, and that line will be perpendicular to the sphere, as represented by OD [so it will be the normal on which the image must lie]. Hence, the

image of any point on line OH will appear inside the sphere [where the line of reflection GH intersects normal OD].

[CASE 2]

[2.85] Now, the arc on the circle lying between point H and the point where the normal dropped from the center of sight passes through [the circle] is HZ. I say that, from whatever point on this arc reflection occurs, the image-location will lie inside the sphere.

[2.86] [Here is] the proof. Let I [figure 5.2.9a, p. 224] be the point selected, draw line GIS from the center of sight to intersect the circle at that point, and draw normal DIT from that point. Make line PI form an angle [PIT] with [normal] IT that is equal to angle TIG. It is clear that only points on line PI are reflected from point I to the center of sight. It is also clear that line IS > line KH [by Euclid, III.8], so it is longer than [line] SD [which = line KH, by construction]. Therefore, angle SDI > angle SID [by Euclid, I.19], so it is greater than angle GIT [which is vertical to angle SID], and therefore greater than angle TIP.

[2.87] Hence, line PI and [line] SD will never intersect [in the direction of P and S],⁴¹ and the line extended to point D from any point on line PI will intersect line SI inside the sphere, SI being the line of reflection [continued inside the circle]. And any line drawn [to D] from any point on line PI will be perpendicular to the sphere, just as PD is. Moreover, since the image-location lies at the intersection of the normal [dropped] from the point viewed and the line of reflection, the image of any point on line PI will lie inside the sphere. It is therefore evident that the appropriate location of all images [produced by reflection from points] within arc HZ will be inside the sphere, which is what was proposed.

[CASE 3]

[2.88] Moreover, if some point [of reflection] is chosen on arc HB, I say that the image will sometimes lie inside the sphere, sometimes on the surface of the sphere, and sometimes outside the sphere.

[2.89] Choose some point on this arc, let it be N [figure 5.2.9b, p. 224], and from point G draw line GNQ to intersect the circle. Then draw normal DNF, and draw line EN to form with normal [DNF] an angle [ENF] equal to angle FNG. Since line NQ < line KH [by Euclid, III.8], it is also shorter than line QD [since QD = KH, by construction], and so angle QDN < angle DNQ [by Euclid, I.19], and therefore smaller than angle GNF [which is vertical to angle DNQ], and therefore also smaller than angle ENF [which = angle GNF, by construction]. Hence, line EN and [line] DQ will intersect. So let the intersection be at point E. It is obvious that line EQD is normal to the sphere,

and it intersects line of reflection GNQ at point Q, which is a point on the sphere. Therefore, if reflection occurs at point N, the image of point E will appear at point Q, and this point lies on the surface of the sphere.

[2.90] If, however, a point, such as R, is chosen on line EN beyond E, the normal dropped from that point to the center of the sphere, e.g., RD, will intersect line of reflection GNQ beyond point Q [e.g., at point L]. This lies outside the sphere, so the image of any point chosen on line EN beyond E will lie outside the surface of the mirror.

[2.91] But if some point [such as C] is chosen on line EN in front of point E, the normal dropped from it to the mirror [along CD] will intersect line GNQ inside the sphere, because [it intersects it] at a point that lies between N and Q. Accordingly, the image of any point chosen on line EN between E and N will appear inside the sphere.

[2.92] The very same proof will hold for any other point chosen on arc BH. And so, for any point on arc BH, one image alone lies on the surface of the mirror, others lie inside the mirror, and others lie outside it. Moreover, what has been demonstrated for arc ZB can be shown in exactly the same way for arc ZA, and the very same demonstration will apply for any [great] circle on the sphere and the portion [of it] that is selected to face the center of sight and be bisected by normal GD.

[2.93] Accordingly, if the center of sight remains immobile with normal GD fixed in place, and if line GHK is rotated uniformly about normal [GD as axis], it will cut off a circular portion of the sphere with its rotation, and the image of any point [reflected from any point] within this portion will appear inside the sphere.⁴²

[2.94] On the other hand, if line GB tangent [to the sphere] is rotated uniformly about normal [GD as axis], it will cut off a portion of the sphere larger than the previous one, and from any point on the portion that constitutes the excess of the latter over the former, [one] image reflected [to the eye] will find its location on the surface of the sphere, and of the remaining images, some [will lie] inside the sphere, and some outside it.⁴³

[2.95] From these facts, we know that in this [sort of] mirror any image appears on a diameter of the sphere [that forms the normal, and that image may appear] inside the sphere, outside it, or on its surface. Moreover, any diameter upon which an image may appear, whether on the surface of the sphere or outside that surface, lies below the point on the sphere where the tangent extended from the center of sight to the sphere touches the sphere, that point being at the very edge of the visible portion [of the mirror]. We [also] know [from this analysis] that every line of reflection cuts the sphere in two points, the point of reflection and some other one.⁴⁴

[2.96] **[PROPOSITION 10]** It remains for us to determine image-locations more specifically. I say that, if a diameter is chosen [on a convex spherical mirror], and if a line is extended to it from the center of sight to cut the sphere so that the segment lying between the point of intersection on the sphere and the point on the diameter to which it reaches is equal to the segment of the diameter lying between that latter point and the center [of the sphere], that point does not constitute the location of any image.

[2.97] For instance, let AG [figure 5.2.10, p. 225] be a [great] circle on the sphere, H the center of sight, and ED a diameter, or normal, in the sphere. Let HZ be a line intersecting the sphere at point F and meeting ED at point Z, and let $ZF = ZD$. I say that Z does not constitute the location of any image [on normal DE].

[2.98] For it is evident that there is no image-location at any point other than on ED, because the image of any point lies on the diameter extended from it to the center of the sphere. That the image-location of any point on ED does not lie at Z will be demonstrated as follows.

[2.99] From point D draw normal DFN to point [of reflection] F, and at point F form angle QFN equal to angle NFH. It is therefore evident that angle QFN = angle ZFD [which = vertical angle NFH]. But [angle] ZFD = angle ZDF, because they are subtended by equal sides [of isosceles triangle ZFD, sides ZF and ZD being equal, by construction]. Therefore, [angle] QFN = angle ZDN, so [by Euclid, I.27] line [of incidence] FQ is parallel to line ED [which forms the normal].

[2.100] Hence, if they are extended indefinitely, they will never meet. Accordingly, the form of no point on [normal] ED will pass to point F along [line of incidence] QF, and there can be no image-location for any point at point Z unless its form passes to F along line QF.⁴⁵ The same disproof will hold for any diameter that is chosen, so what was proposed [is demonstrated].

[2.101] I say, as well, that no point on line ZD can be the location for any image.

[2.102] For let point P be taken [on line DZ], and draw line [of reflection] HP to intersect the sphere at point [of reflection] B. Then draw normal DBM, and make angle [of incidence] TBM = angle [of reflection] MBH. It is clear that [angle] TBM = [angle] PBD [which = vertical angle MBH], and it is clear that angle DPH > angle PZF, because it is exterior [to triangle PZH]. Therefore, the two remaining angles [PDB + DBP] of triangle DPB < the two remaining angles [ZDF + DFZ] of triangle DZF [since angle PZH is common to both triangles PZH and DZF]. But angle PDB > angle ZDF. It therefore follows that angle DBP < angle DFZ. But angle DFZ = angle ZDF [since triangle DZF is isosceles by construction], so angle DBP < angle ZDF. Hence,

it is even smaller than angle PDB [since we have already established that angle PDB > angle ZDF], so angle [of incidence] TBM [which = angle DBP] < angle PDB. Therefore, lines TB and ED will never intersect [on the side of T and E], and so no image [created by reflection] from point B is produced at point P. The same holds for the image of any other point [on line of incidence TB], and likewise for any point on line ZD. It follows, therefore, that the whole of ZD is void of image-locations.

[2.103] **[PROPOSITION 11]** Furthermore, if a diameter, other than the one imagined [to extend] from the center of sight to the center of the sphere, is taken between the lines drawn tangent to the sphere from the center of sight, and if the point on it we [just] described as the limit of image-locations is determined, I say that the image-locations of points on that diameter lie only at points on it between the surface of the sphere and the aforementioned limit.

[2.104] For example, let BZ and BE [figure 5.2.11, p. 225] be the tangents, B the center of sight, A the center of the sphere, BHA the visual axis [which is normal to the mirror from the center of sight], and DA the selected diameter with G [the point at which DA intersects the sphere and T] the limit [of image-locations].⁴⁶ I say that images of points [whose forms are reflected to B] occur only at points between G and T on DA.

[2.105] That they will not occur at point G or [any point] outside the sphere's surface is clear from what was said above [in proposition 9], i.e., that the diameter on which the image-location will occur at the surface of the mirror or outside it lies below the point of tangency [E]; and since diameter DA lies between the lines of tangency, there will be no image-location on it either at or outside the [mirror's] surface. On the other hand, it will be demonstrated that the image may fall at any point taken between G and T.

[2.106] Choose [such] a point, let it be Q, and draw line [of reflection] BQ to cut the sphere at point C. Now, draw normal ACL, make angle [of incidence] DCL = angle [of reflection] LCB, draw line BT to cut the sphere at point F, and draw normal AF. Accordingly, triangle ACB contains triangle AFB, so angle AFB > angle ACB. It follows, then, that angle AFT [adjacent to larger angle AFB] < [angle] ACQ [adjacent to smaller angle ACB]. But angle AFT = angle FAT, because they are subtended by equal sides [of isosceles triangle FAT]. Therefore [angle] ACQ > angle CAQ [because angle CAQ < angle FAT < angle ACQ], so [angle] LCB [which = vertical angle ACQ] > [angle] CAQ, from which it follows that [angle] DCL > [angle] CAQ [since angle DCL = angle LCB, by construction]. CD and AQ will therefore intersect. Let D be the [point of] intersection. Thus, the form of point D should reflect at point C along line CB, and its image-location is Q. And the same proof holds for any [other] point selected between G and T.

[2.107] **[PROPOSITION 12]** It remains for us to determine image-locations in the invisible section of the sphere.

[2.108] Accordingly, let AC and AG [figure 5.2.12, p. 225] be the tangents [defining] the visible portion [of the sphere], A the center of sight, B the center of the sphere, ADBZ the visual axis [which is normal to the mirror from the center of sight], and ZCG the [great] circle on the sphere lying in the plane of the tangent lines. From the center [of the sphere] draw diameter BG to point [G] of tangency. It is evident that angle ZBG > a right angle, for, since angle BGA in triangle BAG is right, angle GBA < a right angle, so [adjacent angle] ZBG will be greater [than a right angle]. Accordingly, let HBG be a right angle. HB will thus be parallel to line of tangency AG. Hence, when extended [indefinitely] they will never intersect, whereas any diameter between H and G will intersect line AG.

[2.109] Draw line AMO from point A to intersect the sphere so that chord MO = radius OB, and let diameter BO intersect line AG at point T. I say that there is an image-location at any point on TO, that there is an image-location at no other point on diameter TB, and that O and T are the limit-points for image-locations [on that diameter].⁴⁷

[2.110] Take [such a] point, let it be K, and draw [line of reflection] ANK to cut the sphere at point N. Then draw normal BNY, and form angle [of incidence FNY] with FN that is equal to angle [of reflection] YNA. It is clear that [line of incidence] FN will not lie between B and T, because [in that case] it must cut either the sphere or tangent AP in two points.⁴⁸ Therefore, the form of point F will propagate along FN to point N and will be reflected to A along line AN, and its image will appear at point K. And the same proof holds for any other point chosen [as an image-location between T and O].

[2.111] **[PROPOSITION 13]** I say, further, that no matter what diameter may be chosen within arc OG, it will contain image-locations inside the mirror, as well as one [image] on the surface of the mirror, and other [images] outside the mirror.

[2.112] Accordingly, take point L [in figure 5.2.13, p. 226], and draw diameter BL until it intersects [line of tangency] AP at point E. Then draw line AL to intersect the sphere at point R. It is obvious [by Euclid, III.8] that $RL < LB$, since it is shorter than MO, which is equal to the [sphere's] radius [by construction from the previous proposition]. Accordingly, if from A a line is drawn to diameter LB such that the segment [of it] lying between the circle and the diameter is equal to the segment of the diameter [extending] from the point where that line meets the diameter to the center [of the circle], it will fall between L and B. For if it were to fall between L and E, then $RL >$

LB, and every line falling between the center [of the sphere] and that equal segment [on LB] would be longer than the segment on the diameter where it terminates, according to the proof [in proposition 10] devoted to explaining the limit of image[-locations].⁴⁹

[2.113] So let I be the point where the line[-segment between arc DG and normal BL] equal [to the segment on BL] will fall. I say that there is an image-location at any point on EI [other than E or I]. And the same demonstration will apply [in this case] as applied for TO [in the previous theorem].

[2.114] Accordingly, the image-locations on diameter EB are divided among those that lie inside the mirror [i.e., between L and I], those that lie outside the mirror [i.e., between E and L], and the single one, i.e., at point L, that lies on the mirror's surface. And in this way you can demonstrate [the same thing] for any diameter that passes through arc OG.

[2.115] **[PROPOSITION 14]** Moreover, if some diameter is taken within arc OH, [any] image-location on that diameter will lie outside the mirror.

[2.116] Take diameter BQ [figure 5.2.14, p. 227], and let it intersect the tangent [AG] at point P. Then draw line ANQ to cut the sphere at point N. It has already been established [by construction in propositions 12 and 13] that $MO = [\text{radius}] OB$, but $NQ > MO$, so $NQ > [\text{radius}] QB$. Moreover, the line drawn at the circumference [of the sphere] to diameter PB and equal to the segment of BP lying between it and the center [of the sphere] will not fall between Q and B. For if it were to fall [there], then, according to the previous demonstration, $NQ < QB$ [which contradicts the conclusion drawn above].

[2.117] It follows, therefore, that the line equal [to the previously defined line] should fall between P and Q. That it may not fall at point P is clear from the fact that angle PGB is a right angle [and so tangent AGP cannot be a line of reflection]. Therefore, $PB > PG$ [because it subtends a larger angle in triangle BGP]. So it will fall in front of P [i.e., between P and the mirror].

[2.118] Let the point at which it falls be S [so that the line-segment on AS between S and the point at which AS cuts the circle is equal to SB]. S will therefore constitute the limit for image-locations, and any point between P and S will constitute an image-location, and the proof for this is the same as above.

[2.119] From these theorems it is evident that all the images [produced] on diameters within arc HO lie outside [the mirror]; on diameter FB, one lies on the surface, at O; and all the rest lie beyond, i.e., in TO; however, of all the images on [any] diameter within arc OG, some lie inside [the mirror], some outside it, and one on its surface.

[2.120] **[PROPOSITION 15]** Furthermore, no diameter can be selected in arc HZ [figure 5.2.15, p. 227] that contains an image-location, for no diameter selected there intersects tangent AP [since HB is parallel to AP].

[2.121] Now, from some point on such a diameter [outside the sphere], draw a line [tangent] to the sphere. It will of course touch [the sphere] within segment GZC, not within segment GDC, unless it cuts the sphere. Accordingly, no form of any point on such a diameter will reach the portion [of the mirror] visible to the eye.

[2.122] Moreover, what has been claimed for arc GH can be demonstrated in the same way for the portion of arc CZ that corresponds to it. So if an arc equal to arc HZ is taken on the other side of Z, there will be no image-location on any diameter within that arc.

[2.123] The same method is used in proving this for any circle [on the sphere], so if line HB is rotated [about axis BZ] while angle HBZ is kept constant [throughout its rotation], it will describe with its motion a portion of the sphere within which there is no diameter with an image-location. If, however, HB is held in place [so that angle HBZ remains constant], and if OB is rotated [about axis BZ], it will describe a portion [of the sphere] within which all of the images [on any diameter lie] outside [the surface, by proposition 14], but on diameter TB one [image lies] on the surface, the rest outside [by proposition 12]. Finally, if arc OG is rotated, it will form a portion [of the sphere] within which some of the images lie on the surface, some outside the sphere, and some inside [by proposition 13].

[2.124] Yet the visual faculty does not perceive which images lie on the surface of the sphere or which [lie] outside, nor does it determine anything in the process of perceiving them except that they lie behind the visible portion [of the sphere].⁵⁰ At this point, then, the image-locations in these [sorts of] mirrors have been determined.

[2.125] **[PROPOSITION 16]** Furthermore, in this [sort of] mirror, the form of a visible point can be reflected to the eye from only one point on the mirror.

[2.126] Let the [visible] point be B [figure 5.2.16, p. 228], let A be the center of sight, and let A not stand on the normal dropped to the center [of the sphere through the point of reflection]. I say that [the form of] B is reflected to A from only one point on the mirror, and it yields only one image to the eye in this [kind of] mirror.

[2.127] It is clear that its form can be reflected from some point. Let that point be G, and draw [line of incidence] BG and [line of reflection] AG. Let N be the center of the sphere, draw diameter BN [i.e., the normal dropped from object-point B] to intersect the sphere's surface at point L, and let the

limits of the [visible] portion [of the mirror] facing the center of sight be D and E [that portion being subtended by tangents AE and AD]. Let line AG intersect normal [BN] at point Q, which is the image-location.

[2.128] It is obvious that A, N, and B lie in the same plane orthogonal to the sphere. And since all planes that are orthogonal to the sphere and that contain BN intersect along BN, and since only one plane containing line BN can be extended through point A, it is clear that A and B lie in only one plane that is orthogonal to the sphere, not in several [such planes]. Moreover, since the visible point [B] and [center of sight] A must lie in the same plane that is orthogonal [to the sphere] at the point of reflection, it is clear that the reflection of [the form of] point B to the eye will occur only on the [great] circle that lies in plane ANB within the sphere. Accordingly, let that circle be DGE. I say, again, that reflection will occur from no point other than G on this circle.

[2.129] If it is claimed that [such reflection occurs] from point L [on normal BN], then, since BN is orthogonal [to the sphere] while AL is not orthogonal, and since the form that is incident along the perpendicular is necessarily reflected back along the perpendicular, it is obvious that [the form of] B does not reflect to A from point L. Nor, by the same token, [does it reflect] from another point on arc LE. For, to whatever point on that arc a line is dropped from point B, it will form an obtuse angle with the tangent at that point on the side of E, whereas the line dropped from point A to that point will form an acute angle with that tangent on the side of L. Therefore, if reflection were to occur from that point, an acute angle would equal an obtuse angle.

[2.130] Likewise, reflection cannot occur from any point on [arc] GL. Take some [such] point, let it be Z, and draw line AZO to intersect normal [BN] at point O. Then draw the line tangent to the circle at point Z, a line that necessarily falls between [i.e., intersects] BG and BL, and let it be MZ. Let FG be the line tangent to the circle at point G. It is clear from the above that $BN:NQ = BF:FQ$ [from proposition 7 above, F being the endpoint of tangency]. So too, $BN:NO = BM:MO$ [from proposition 7 above, Z being the point of reflection and M the endpoint of tangency]. But $BN:NQ > BN:NO$ [by Euclid, V.8]. Therefore $BF:FQ > BM:MO$, which is clearly impossible, since $BF < BM$, while $FQ > MO$. It therefore follows that reflection may not occur from point Z.

[2.131] On the other hand, it will be proven as follows that reflection may not occur from any point on arc GD. Take some [such] point, and let it be T. Draw line BT, and [draw] line ATH to intersect BN at point H. Then draw CT tangent to the circle at point T. Accordingly, $BN:NH = BC:CH$ [by proposition 7 above, T being the point of reflection and C the endpoint of

tangency], and $BN:NQ = BF:FQ$ [as established above]. But $BN:NH > BN:NQ$ [by Euclid, V.8]. Hence, $BC:CH > BF:FQ$, which is clearly false, since $BF > BC$, while $CH > FQ$. It therefore follows that reflection of [the form of] point B may occur from no point on arc GD, so [the form of] any given point is reflected to the eye from only one point on the sphere. Hence, there will be only one line of reflection for each visible point, and so [there is] only one image of [any] single point.

[2.132] Moreover, if point B lies on the visual axis [which is the normal dropped from the eye to centerpoint N of the sphere], it is obvious that it is reflected from only one point, because [it is reflected] only along the normal, and its image will be unique and will lie in the appropriate image-location because of its continuity with the other points [as established in proposition 8].

[2.133] **[PROPOSITION 17]** Furthermore, if two points are taken on a given diameter on the same side of the [sphere's] center, the image-location of the point nearer the [sphere's] center will lie farther from the center of the sphere than the image-location of the point farther from the sphere's center. Also, the point of reflection for the point nearer the [sphere's] center will lie farther from the center of sight than the [point of reflection] for the point farther from the center of the sphere.

[2.134] I say, for example, that the image-location for point C [figure 5.2.17, p. 229] lies farther from the center [N] than does the image-location for point B, and that the point of reflection for point C lies farther from point A [the center of sight] than the point of reflection for point B, i.e., point G. I say [further] that [the form of] point C is reflected [to A] from a point on arc GL only.

[2.135] For it is evident [from the previous theorem] that it will not be reflected from any point on arc LE, nor from point L, nor from point G, since [the form of] B is reflected from that point. But if it is claimed that [it is reflected] from some point on arc GD, let that point be T, and let CT be the line along which the form [of C] radiates to the mirror. Then draw normal NT, which will necessarily bisect angle CTA, and draw normal NGK. Angle $NTA > [\text{angle}] \text{NGA}$ [by Euclid, I.21]. It therefore follows that angle PTA [adjacent to angle NTA] $< \text{angle KGA}$ [adjacent to angle NGA], so angle CTP $< \text{angle BGK}$ [because, by supposition, angle CTP = angle PTA, and angle BGK = angle KGA, and we have just concluded that angle PTA $< \text{angle KGA}$]. But [by Euclid, I.32] angle CTP = angle TNC + angle TCN, because it is an exterior [angle of triangle TCN]. In addition, angle BGK [which is an exterior angle of triangle BGN] = angle GNB + angle GBN. Accordingly, the [sum of the] two angles TNC and TCN $< \text{the [sum of the]}$

two angles GBN and GNB, which is impossible, since angle TNC contains [angle] GNB as a part, while angle TCN > [angle] GBN.

[2.136] Hence, it follows that point C may only be reflected from [some point among the] points lying between G and L. All lines drawn from point A to normal BN through such points fall on points [between L and Q] that lie farther than point Q from the center of the sphere, and they fall to points on the sphere [between L and G] that are farther away from the center of sight [A] than point G, and so what was proposed [has been proven].

[2.137] **[PROPOSITION 18]** Now, given a [convex spherical] mirror and given a visible point, to find the point of reflection.

[2.138] Let B [figure 5.2.18, p. 229] be the visible point, [let] A [be] the center of sight, and draw two lines from these points to the center of the mirror. If those lines are equal, it will be easy to find [the point of reflection]. For a [great] circle on the sphere will be defined in the plane within which those two lines lie, and we know that reflection occurs from only one point on that circle. Accordingly, the angle those two lines form at the [sphere's] center will be bisected.

[2.139] Draw the line [GN] that bisects the angle [and extend it] beyond the sphere. It will of course be perpendicular to the line tangent to this circle at the point [G] through which it passes. And if two lines [AG and BG] are drawn to that point, one from the center of sight and the other from the visible point, they will form two triangles with that normal and the two original lines, two sides of these triangles [i.e., BN, NG and AN, NG] being equal [respectively] and angle [BNG of the one, being equal] to angle [ANG of the other].⁵¹ Hence, the point on the circle through which that normal passes constitutes the point of reflection, which is what was set out [to be demonstrated].

[2.140] On the other hand, if the line extending from the visible point to the center of the sphere is not the same length as the line extending from the center of sight to that same centerpoint, we must set forth some preliminary things, one of which is as follows.

[2.141] **[PROPOSITION 19, LEMMA 1]** If a diameter is taken in a circle, and if a point is taken on its circumference, a line can be drawn from that point to the extension of the diameter beyond [the circle] such that its extension from the point where it intersects the circle to the point where it intersects the diameter is equal to a given line.

[2.142] For instance, let QE [figure 5.2.19, p. 230] be the given line, GB the diameter of circle ABG, and A the given point [on its circumference]. I say that I can draw a line from point A such that [the segment extending]

from the point where it intersects the circle to diameter GB is equal to line QE, which will be proven as follows. Draw the two lines AB and AG, which are either equal or unequal.

[CASE 1]

[2.143] Let them be equal, and add a line to QE—let EZ be this additional line—such that the rectangle that will be formed by the whole line augmented by the additional line and the additional line [i.e., $(QE + EZ) \cdot EZ = QZ \cdot EZ = AG^2$].⁵² Accordingly, since $QZ \cdot EZ = AG^2$, $QZ > AG$. For if EZ were equal to, or longer than AG, it is impossible that $QZ \cdot EZ = AG^2$ [because the whole, QZ, is greater than its part, EZ, whose square = AG^2]. On the other hand, if it is shorter, it is clear that $QZ > AG$.

[2.144] Accordingly, extend AG [figure 5.2.19a, p. 230] until it is equal [to QZ], and let the result be AGT. Then, placing a compass-point at A, draw a circle according to radius AGT, and [since] this circle will intersect diameter BG, let it intersect [it] at point D. Then draw line AD, which will necessarily intersect circle [AGB], for if it were tangent at point A, it would be parallel to BG and would never intersect it. Accordingly, let it intersect [the circle] at point H, and draw line GH.

[2.145] It is clear that, since ABGH constitutes a quadrilateral within the circle, the two opposite angles, i.e., ABG and AHG, sum up to two right angles [by Euclid, III.22]. But [angle] AGB = [angle] ABG, since they are subtended by equal sides, according to construction. Therefore, angle AHG = angle DGA [because, as previously concluded, angle AHG = two right angles – angle ABG, and angle DGA = two right angles – angle AGB, which = angle ABG], and angle HAG is common to the whole triangle ADG and triangle AHG that forms part of it. It therefore follows that angle HDG = angle HGA and that [the one] triangle [DGA] is similar to [the other] triangle [AHG], so [by Euclid, VI.4] $DA:AG = AG:AH$. Hence, $DA \cdot HA = AG^2$ [by Euclid, VI.17]. But $DA = TA$ [since they are both radii of circle TD], so $DA = QZ$ [since $TA = QZ$, by construction]. And [therefore] $AH = EZ$, and $DH = QE$, which is the given line, and so what was proposed [has been demonstrated].⁵³

[CASE 2]

[2.146] Now, if AB and AG are not equal [figure 5.2.19c, p. 231], draw a line from point G that is parallel to AB, let it be GN, and take some line ZT, and form an angle at point Z with line ZF [i.e., angle TZF] equal to angle AGD. Then, from point T draw a line parallel to ZF, let it be TM, and from angle TZF cut off an angle with line ZM [i.e., angle MZT] equal to angle NGD, for this line [MZ] will necessarily intersect TM, since it lies between

parallels [TM and ZF]. Let the point of intersection be M. It follows, therefore, that angle MZF = angle AGN [since, by construction, angle TZF = angle AGD, and angle TZM = angle NGD, so angle MZF (which = angle TZF – angle TZM) = angle AGN (which = angle AGD – angle NGD)].

[2.147] Now, from point T draw a line TO parallel to line ZM, and this line will necessarily intersect FZ. Let the point of intersection be K. Take some line I that is to line ZT as [line] BG is to [line] EQ, [which is the] given line [i.e., $I:ZT = BG:EQ$, by Euclid, VI.12]. Then, at point M produce a conic section [i.e., a hyperbola] in the way Apollonius describes in the fourth proposition of book 2 of his *Conics*, and let this conic section be UCM, which may not intersect lines KO and KF [that form its asymptotes].⁵⁴ In this section draw a line equal to line I, i.e., MC, extend it to lines KT and KF, and let O and L be the intersection-points [where MC extended cuts asymptotes KT and KF]. Thus, as the same [book of Apollonius' *Conics*] will demonstrate, $OM = CL$.⁵⁵

[2.148] From point T draw line TF parallel to [line] CM, and at point A form an angle [GAN] with line AND that is equal to angle ZFT. It is evident that this line [AND] will intersect GD, since angle AGN = angle FZM, and angle GAN = angle ZFT. Hence, line AD will either be tangent to the circle or will intersect it, because, if it is not tangent, and if arc AB > arc AG, it will cut arc AB, whereas if $AB < [AG]$, it will cut arc AG.

[SUBCASE 2A]

[2.149] Accordingly, let it be tangent at point A. Since angle GAN [in triangle GAN] = angle ZFT [in triangle ZFY], and since angle AGN = angle FZY, the third angle [ANG] = the third angle [ZYF], so [by Euclid, VI.4] triangle AGN will be similar to triangle ZFY. Likewise [in triangles AGD and FZT], since [angle] AGD = angle FZT [while angle ZFT = angle GAD, and the remaining angles ZTF and ADG are equal], then triangle AGD will be similar to triangle FZT. Therefore $AN:AG = FY:FZ$, and $AG:GD = FZ:ZT$, so $AN:GD = FY:ZT$ [by Euclid, V.22].

[2.150] But since TM is parallel to FL [by construction], and since FT is parallel to ML [by construction], $FT = ML$, so it will be equal to CO, since $MO = LC$ [by Apollonius, II.8]. But $MO = YT$, since it is parallel to it, and YM is parallel to TO. It therefore follows that $FY = CM$. But $CM = I$ [by construction]. Thus, $FY = I$. But $I:ZT = BG:EQ$ [by construction]. Therefore, $AN:GD = BG:EQ$ [because we established earlier that $AN:GD = FY:ZT$].

[2.151] However, angle GAN = angle GBA, as Euclid demonstrates in the third [book of the *Elements*, prop. 32]. But angle NGD = [alternate] angle ABG, since NG is parallel to AB [by construction]. Therefore, angle NGD [in triangle NDG] = angle NAG [in triangle AGD], and angle NDG is com-

mon [to both triangles], so the third [angle, i.e., DNG] = the third [angle, i.e., AGD], and so triangle NDG is similar to triangle ADG. Hence, $AD:GD = GD:ND$, so $AD, DN = DG^2$ [by Euclid, VI.17].

[2.152] But $AD^2 = BD, DG$, as Euclid demonstrates [in III.36], and $AD^2 = AD, DN + AD, NA$ [by Euclid, II.2]. And $BD, DG = DG^2 + BG, GD$, as Euclid demonstrates [in II.3]. Therefore, when equal terms are subtracted [i.e., GD^2], it follows that $AD, AN = BG, DG$.⁵⁶ Hence [by Euclid, VI.16], the second [AN] is to the fourth [DG] as the third [BG] is to the first [AD], so $AN:DG = BG:AD$. But it has already been established that $AN:GD = BG:EQ$. Therefore, $EQ = AD$, which is what was proposed.

[SUBCASE 2B]

[2.153] If, however, AD is not tangent to the circle but cuts it, and if $AG > AB$ [figure 5.2.19d, p. 232], then it will cut [arc] AG. Let it cut at point H, and draw line AG.

[2.154] It is obvious that the two angles AHG and ABG sum up to two right angles [by Euclid, III.22]. But angle NGD = [alternate] angle ABG [because NG is parallel to AB, by construction]. Thus, angle AHG + angle NGD = two right angles. Accordingly, angle NGD = angle NHG [since angle AHG + angle NHG = two right angles], and angle NDG is common [to triangles NGD and HGD], so the third angle [DNG] = the third angle [DGH], and so triangle HGD is similar to triangle NDG. Therefore, $HD:DG = DG:DN$, so $HD, DN = GD^2$ [by Euclid, VI.17].

[2.155] But $AD, DH = BD, DG$, as Euclid demonstrates,⁵⁷ and $AD, DH = DH, DN + DH, AN$ [by Euclid, II.1]. Moreover, $BD, DG = BG, GD + GD^2$ [by Euclid, II.3]. Therefore, when equal terms are subtracted (i.e., GD^2 and DH, DN), it follows that $DH, AN = BG, DG$,⁵⁸ so [by Euclid, VI.16] the second term is to the fourth (i.e., $AN:GD$) as the third is to the first (i.e., $BG:DH$). But it has already been proven that $AN:DG = BG:EQ$. Thus, $EQ = DH$, and so what was proposed [has been demonstrated].

[SUBCASE 2C]

[2.156] On the other hand, if $AG < AB$ (and let [AD] cut [the circle] on arc AB), then let H [figure 5.2.19e, p. 232] be the point of intersection, and draw line HG. It is evident that angle NGD = angle ABG [by construction]. But [by Euclid, III.21] angles ABG and AHG are equal, since they are subtended by the same arc [AG]. Therefore, angle NGD = angle AHG, and angle NDG is common [to triangles NGD and HGD], so the third angle [GND] = the third angle [HGD], and the triangles [NGD and HGD] are similar. Accordingly $HD:GD = GD:DN$, so $HD, DN = GD^2$ [by Euclid, VI.17].

[2.157] But $HD, DA = BD, DG$ [since, by Euclid, III.36, both are equal to the square on the tangent drawn from D], and $HD, DA = DN, HD + AN, HD$

[by Euclid, II.1]. Moreover $BD \cdot DG = DG^2 + BG \cdot DG$ [by Euclid, II.3]. Hence, with equal terms subtracted [i.e., GD^2 and $HD \cdot DN$], $HD \cdot NA = BG \cdot DG$. Accordingly, $AN : DG = BG : HD$ [by Euclid, VI.16]. But it has already been established that $AN : DG = BG : EQ$. Therefore, $EQ = HD$, which is what was proposed, because from point A we have drawn a line to cut the circle, and [the segment] from the point of intersection [H] to the diameter [BG] is equal to the given line.

[2.158] **[PROPOSITION 20, LEMMA 2]** Now, from a given point on a circle outside its diameter, a line can be drawn through the diameter to the circle so that the segment on it between the diameter and the circle is equal to a given line.

[2.159] For instance, let ABG [figure 5.2.20, p. 233] be the given circle, BG its diameter, A the given point, and HZ the given line. I say that a line can be drawn from point A to pass through diameter BG such that the segment [ED] from the diameter to the circle is equal to line HZ.

[2.160] The proof [is as follows]. Draw lines AB and AG, and at point H form with line HM an angle [MHZ] equal to angle AGB, and at the same point form with line HL an angle [LHZ] equal to angle ABG. Then, from point Z draw a line ZN parallel to line HM, and it will intersect HL; and from point Z draw line ZT parallel to HL, and let it intersect HM at point T. At point T construct conic section [i.e., hyperbola] TP, which Apollonius will describe in his book on conics [II.4], and that [conic] section will never touch either of the lines ZN and HL [i.e., the asymptotes] between which it lies. Between these same lines construct conic section [i.e., hyperbola] CU facing the first one.

[2.161] Thus, if the shortest of all the lines extending from point T to [conic] section CU is equal to diameter BG, a circle produced according to this shortest line [as radius] when the point of a compass is placed at point T will be tangent to [conic] section CU. On the other hand, if the shortest of all the lines extending from point T to [conic] section CU is shorter than diameter BG, the circle produced in the aforementioned way according to [radius] BG will intersect [conic] section CU at two points.⁵⁹

[2.162] Accordingly, let CT be the shortest [line], and [let it be] equal to diameter BG, and it will intersect ZQ and HF when it is extended to the [conic] section lying between them.⁶⁰ From point Z draw a line parallel to this one, and it will intersect HM and HL just as its parallel [CT] does. Let it be MZL, let it intersect [those lines] at points M and L, let Q be the point where CT intersects ZN, and on diameter GB form an angle equal to angle HLZ, and let it be DGB. Then draw the two lines AD and BD.

[2.163] It is evident that, since angle GAB is a right angle [by Euclid, III.31], the other two angles in triangle AGB sum up to a right angle, so

angle LHM is right [since it consists of the two angles MHZ and ZHL that were constructed equal to angles AGB and ABG, which sum up to a right angle], and it is equal to angle GDB [which is a right angle by Euclid, III.31]. Also [by construction], angle HLM [in triangle HLM] = angle DGB [in triangle DGB]. Hence, the third [angle HML is equal] to the third [angle DBG], and the [first] triangle [HML] is similar to the [second] triangle [GDB], so $GB:BD = LM:MH$.

[2.164] But, since angle ADB = angle BGA, because they are subtended by the same arc [AB], and since angle BGA = angle MHZ, by construction, then angle ADB = angle MHZ. In addition, we already know that angle GBD = angle HMZ [by the similarity of triangles HML and DGB]. Hence, the third [angle in triangle DEB, i.e., angle DEB, is equal] to the third angle [in triangle MHZ, i.e., angle MZH], so triangle DEB is similar to triangle MHZ. Let E be the point where line AD intersects diameter BG. Accordingly, $BD:DE = MH:HZ$. Hence, $BG:DE = LM:HZ$.⁶¹

[2.165] However, Apollonius demonstrates [in *Conics*, II.16] that, when two [hyperbolic] conic sections lie opposite one another between two [asymptotic] lines, and when a line is drawn from one section to the other, the segment of that line lying between one of the sections and one of the [asymptotic] lines is equal to the opposite segment lying between the other section and the other [asymptotic] line, so $QC = TF$. But $TQ = MZ$, since it is parallel to it [by construction] and lies between two parallels [HM and ZN, which are parallel by construction]. Therefore $MZ = FC$ [since $FC = TQ$], and $ZL = TF$ [since they are parallel and lie between TZ and HL, which are parallel by construction]. Hence, $ML = TC$, so $GB:ED = TC:HZ$, and since $TC = BG$, $ED = HZ$, which is what was proposed.

[2.166] However, if the line extending from T to [conic] section CU is the shortest line and is shorter than diameter BG, then extend it beyond the [conic] section until it is equal. Then, according to its length [as radius], produce a circle that will intersect the [conic] section at two points, and from those points the lines extending to T will be equal to BG. Then, from point Z draw a line parallel to both, and in that case two lines equal to the given lines will be drawn from point A in the prescribed manner, and this will be proven in precisely the same way.⁶²

[2.167] **[PROPOSITION 21, LEMMA 3]** Moreover, given a right triangle, and given some point on one of the sides forming the right angle, a line can be drawn from that point to the other side forming the right [angle] and intersecting the third side facing the right angle in such a way that the segment of this line that lies between the point of intersection and the side on which the given point does not lie is to the segment of the side opposite

the right angle from the [point of] intersection to the side containing the given point as a given line is to a given line.

[2.168] For instance, ABG [figure 5.2.21, p. 235] is the given triangle with ABG the right angle, and the given point D , which is on side GB , is either in or outside the triangle. I say that from point D a line can be drawn to cut side AG and intersect side AB in such a way that the segment $[TQ]$ on it between sides AB and AG is to the segment $[TG]$ of the side AG extending from that line to point G in the same proportion as E is to Z , those being the given lines [i.e., $TQ: TG = E: Z$].

[CASE 1]

[2.169] The proof [is as follows]. Let point D lie on triangle ABG itself [as represented in figure 5.2.21], and draw from it a line DM that is parallel to AB . Produce a circle upon the three points G , M , and D [by Euclid, IV.5], and draw line AD . Since it is evident that angle $GMD =$ [alternate] angle GAB , it will be greater than angle GAD . With line MN cut from [angle] GMD an angle equal [to GAD], and let it be DMN , and [by Euclid, VI.12] draw line H so that AD is to it as E is to Z [i.e., $AD: H = E: Z$]. Then, from point N , which lies on the circle, draw a line to diameter GM ⁶³ that is equal to line H , according to the previous account [in proposition 19, lemma 1], let it be [on line] NL , and let point C be where it intersects the circle [so that $LC = H$].⁶⁴ Draw line GC , and from point D draw a line to point C , a line which, when extended, necessarily intersects the other line [i.e., AB], since it falls between two parallels [AB and DM , which are parallel by construction] and forms with one of them an acute angle [i.e., MDT]. Let them intersect, then, and let Q be the point of intersection.

[2.170] It is evident that angle $GMD =$ angle GCD , because they are subtended by the same arc $[GND]$, and angle $GMD =$ angle GAB [by construction]. It therefore follows that angle $GCQ =$ angle GAQ [because they are adjacent to equal angles, i.e., GCD and GAB]. Let T be the point where DQ intersects AG , and [so] angle GTC [in triangle TCG] = [vertical] angle ATQ [in triangle ATQ , while angle GCT (i.e., GCQ) = angle TAQ (i.e., GAQ), as established above]. Hence, the third [angle TGC] = the third [angle TQA], so triangle ATQ is similar to triangle TCG . Thus, $QT: TG = AT: TC$.

[2.171] But angle $NMD =$ angle TAD [i.e., GAD , by construction], and [it is also equal to] angle NCD [because they are both subtended by arc ND], so [angle] NCD [= vertical angle TCL in triangle TCL , and it also] equals [angle] TAD [in triangle TAD]. In addition, angle CTL is common to both triangles, so the third [angle TLC] = the third [angle

TDA], and [so] the [one] triangle is similar to the [other] triangle, i.e., [triangle] TLC to triangle TAD. Hence, $TA:CT = AD:LC$ [but $TA:CT = QT:TG$, as established above], so $AD:LC = QT:TG$. But $LC = \text{line } H$, and $AD:H = E:Z$. Therefore, $QT:TG = E:Z$, which is what was proposed.

[CASE 2]

[2.172] If, on the other hand, D is taken on that [same] side [that includes the right angle but extends] beyond the triangle [figures 5.2.21a and 5.2.21b, p. 235], then [in figure 5.2.21a] draw [a line] from point D that is parallel to AB , and let it be DM , and extend AG until it intersects DM at point M . Then produce a circle passing through the three points G , D , and M , and draw line AD . Angle GAD [which is exterior to triangle DAM] will of course be greater than [interior] angle GMD . Form an angle equal to it [i.e., equal to GAD], and let it be NMD , and from point N , which is a point on the circle, draw a line equal to line H such that AD is to that line as E is to Z , and let that line be [on line] NCL , which is extended to diameter MG [so that $CL = H$ and, therefore, that $AD:CL = E:Z$]. Let the point of intersection be L .

[2.173] Thus, since angle NMD and angle NCD sum up to two right angles [by Euclid, III.22], and since angle $NMD = \text{angle } TAD$, the two triangles TCL and TAD will be similar.⁶⁵ In addition, since the two angles GCD and GMD are equal [by Euclid, III.21], the two triangles GCT and TAQ will be similar [so that $DT:LT = TQ:TG = AD:CL$], and so $AD:CL$ (which $= H$) $= QT:TG$, and so $E:Z = QT:TG$, which is what was proposed.⁶⁶

[2.174] **[PROPOSITION 22, LEMMA 4]** Now, given two points, i.e., E and D , and given a circle, to find a point on it such that the line tangent to the circle at that point bisects the angle formed by the lines drawn from the aforementioned points to that point.

[2.175] For example, from point E [figure 5.2.22, p. 236] draw line EG to the center of the given circle, extend it to the circumference, and let [this extended line] be ES . Then, draw line GD , and let line MI be cut at point C so that $IC:CM = EG:GD$ [by Euclid, VI.10]. Then bisect MI at point N , and draw perpendicular NO . On point M form with line MO an angle $[OMN]$ that is half of angle DGS . It is clear that it will be smaller than a right angle, whereas angle ONM [will be] a right angle [by construction]. Thus, MO will intersect NO . Let it intersect at point O , and from point C draw a line CKF to the triangle $[NMO]$ such that $KF:FM = EG:GS$ [by proposition 21, lemma 3, case 2]. Then, with line AG , which is extended to the circle, form an angle at point G that is equal to angle MFK , and let it be angle AGE . Finally, draw the two lines AG and DG . I say that A is the point we are seeking.

[2.176] Draw line EA. Accordingly, since [angle] MFK = angle AGE [by construction], and since KF:FM = EG:GA [by construction], because GA = GS [since G is the circle's center, by construction], then triangle AGE will be similar to triangle MFK [by Euclid, VI.4]. Hence, angle FMK = angle EAG, and angle AEG = angle MKF.

[2.177] From point A, then, draw a line that forms with line AE an angle [EAZ] equal to angle NMK, and let it be line AZ, which will necessarily intersect line GE, because KF:FM = EG:GA, and angle GAZ = angle FMC [since angle EAG = angle FMK, and angle EAZ = angle KMN, by construction, so $EAG + EAZ = GAZ = FMK + KMC = FMC$]. Therefore, just as line MO will intersect [line] FK at point F, AZ will intersect GE. Let the intersection occur at point Z, and draw AZ to point Q so that AZ:ZQ = MC:CI [by Euclid, VI.12], and draw line EQ.

[2.178] Then, from point A draw a [line] parallel to EQ, and let it be AT. Angle AQE = [alternate] angle QAT [because QA intersects parallels AT and EQ], and since the two angles ZEA and EAT sum up to less than two right angles [because angle ZEA + adjacent angle AEG = 2 right angles, and $EAT < AEG$], AT will necessarily intersect EZ. Let T be the point of intersection. It is clear that angle AEG = angle MKF [because, as we concluded above, triangles AGE and MFK are similar]. When line EL is drawn from point E perpendicular to AZ, angle AEL = angle MKN, since [by construction] angle EAL [in triangle EAL] = angle KMN [in triangle KMN], and angle ALE = angle MNK, since both are right angles [so the remaining angles AEL and MKN are equal]. It follows, therefore, that angle LEZ = angle NKC [since both are adjacent to equal angles, i.e., $LEG (= AEL + AEG)$ and $NKF (= MKF + MKN)$, respectively], and right angle ELZ = [right] angle KNC [so triangles ELZ and NKC are similar]. It follows that angle EZL = angle KCN. Therefore, [exterior] angle EZQ [of triangle EZL] = [exterior] angle KCI [of triangle NKC].

[2.179] So it is evident that triangle EAG is similar to triangle FMK, triangle EAL is similar to triangle KMN, triangle ELZ is similar to triangle KNC, and triangle EAZ is similar to triangle KMC. Accordingly, AZ:ZE = MC:CK, and QZ:ZA = IC:CM [by construction], and [so, by Euclid, V.22] QZ:ZE = IC:CK, so triangle QZE is similar to triangle ICK, while triangle QLE is similar to triangle IKN. [Accordingly] NM:NI = AL:LQ [since triangles KNM and AEL are similar], and so AL = LQ [since NM = NI, by construction], EQ = EA, angle EQZ [in triangle EQZ] = angle LAT [in triangle ZAT, by construction], and angle EZQ = [vertical] angle AZT. Hence, the third [angle, i.e., ZEQ] = the third [angle, i.e., ATZ], and triangle EZQ is similar to triangle ZAT, so QZ:ZA = EZ:ZT = EQ:AT = AE:AT [since AE and EQ are equal by previous conclusions]. But QZ:ZA = EG:GD [because EG:GD

= IC:CM, by construction, and QZ:ZA = IC:CM, by construction]. Therefore AE:AT = EG:GD.

[2.180] Now, at point A form an angle equal to angle GAE, and let it be UAG. It is clear that angle GAL is half of angle UAT,⁶⁷ but it is [also] half of angle DGU [since, by construction, OMN is half DGU, and GAL = OMN], so angle UAT = angle DGU. But angles TAU and TUA sum up to less than two right angles, since AT and UT intersect, so the two angles TUA and DGU sum up to less than two right angles. Therefore, AU will intersect DG.

[2.181] I say that it will intersect at point D, because [by Euclid, VI.4] it will form a triangle with lines UG and GD that is similar to triangle AUT, for they will have angle AUG in common, and angle TAU = angle UGD [by previous demonstration]. Therefore AU:AT is as UG is to the line [X] that AU cuts from GD [i.e., AU:AT = UG:X], and EA:AU = EG:GU [by Euclid, VI.3], since angle UAG = angle GAE [by construction].

[2.182] Therefore, since EA:AT = EG:GD [by previous conclusions], EA:AT is compounded from EA:AU and AU:AT [i.e., EA:AT = (EA:AU):(AU:AT)].⁶⁸ EG:GD will be compounded from these same ratios [i.e., EA:AU and AU:AT, so EG:GD = (EA:AU):(AU:AT), but AU:AT = UG:X, by previous conclusions] so it will be compounded from EG:GU and GU:X [i.e., EG:GD = (EG:GU):(GU:X)]. But it is compounded from EG:GU and GU:GD [i.e., EG:GD = (EG:GU):(GU:GD)]. Therefore the line that AU cuts off from GD is line GD [i.e., GD = X]. Therefore AU cuts GD at point D.

[2.183] Accordingly, from point A extend tangent AH. GAH will therefore be a right angle. But [angle] GAL is half of angle DGU [by previous conclusions]. Hence, angle LAH is half of angle DGE, since these two [i.e., DGU and DGE] sum up to two right angles [so their halves, GAL and LAH, must sum up to a right angle—i.e., GAH, which is right by construction]. But since angle TAU = angle DGU, angle TAD [adjacent to TAU] = [angle] DGE [adjacent to DGU]. Therefore, angle LAH is half of angle TAD [because LAH is half of DGE, by previous conclusions], and angle EAL is half of angle EAT [by previous conclusions]. Thus, angle EAH [which = EAL + LAH] is half of angle EAD, so AH bisects angle EAD, which is what was proposed.

[2.184] On the other hand, if the angle at point A [i.e., UAG] = angle GAE, and if AU does not fall on line ES either outside or inside the circle, then let it be parallel [figure 5.2.22a, p. 236]. Accordingly, [alternate] angle UAG = [alternate] angle AGE. But the same angle [i.e., UAG] = angle GAE [by supposition], so angle GAE = angle AGE. Thus EG = AE. Likewise, angle TAD = angle ATG, since they are alternate. But it has already been established that angle TAD = angle DGT [by previous conclusions]. Therefore, angle ATG = angle DGT, and by the same token the two angles ADG

and DGT are equal. Therefore, the two angles ADG and TAD are equal [as are angles ADG and ATG, so triangle AMD is similar to triangle GMT, and both triangles are isosceles because of the equality of the angles at points A and D and at points G, and T].

[2.185] From these conclusions it will therefore follow that the line AU cuts from DG is equal to line AT [i.e., $GM = MT$, and $AM = MD$, so $AM + MT = GM + MD$]. And it has already been established that $EG = AE$. Hence, EG is to the line AU cuts from DG [i.e., X] as AE is to AT [i.e., $EG:X = AE:AT$]. But it has already been established that $AE:AT = EG:GD$. Therefore the line that AU cuts off from DG is GD, and since [angle] TAD = angle DGT, [angle] LAH will be half of angle TAD, as was claimed above, and [angle] EAL [will be] half of [angle] EAT [by previous conclusions]. Accordingly, [angle] EAH will be half of angle EAD, which is what was proposed.

[2.186] **[PROPOSITION 23, LEMMA 5]** Moreover, given a circle with G as its center [figure 5.2.23, p. 237], given GB as a diameter within it, and given point E outside the circle, a line can be drawn from point E to diameter GB that cuts the circle in such a way that the segment of that line [extending] from the circle to the diameter [i.e., DZ] is equal to the segment of the diameter between that line and the [circle's] center [i.e., ZG].

[2.187] For instance, from point E draw EC perpendicular to the diameter, and draw line EG. Take a line QT equal to EC, and on QT form a segment of a circle such that any angle within that segment [e.g., QPT] is equal to angle EGB [by Euclid, III.33], and then complete the circle. From the midpoint [L] of QT extend a perpendicular [FL] in both directions to the circle. This will of course be a diameter of this circle. Then, from point Q draw a line to this diameter to intersect it at point F, and extend it to point P on the circle such that FP is half of GB [by proposition 20, lemma 2], and draw line PT and line TF. From point P draw line PU parallel to the diameter. Let it intersect TF at point U, and from point U draw UO parallel to TQ. From point T draw TN orthogonal to PQ, from point T draw TS parallel to PQ, and from point U draw UH orthogonal to PQ. Then, from angle BGE cut off an angle BGD equal to angle QPU, and draw line EDZ. I say that $DZ = ZG$.

[2.188] Now, from point D draw DI orthogonal to BG, and from point D draw tangent DK. It is clear that, since diameter FL is perpendicular to QT as well as to OU, and since PU is parallel to that diameter, angle OUP will be a right angle. And since OU is bisected by the diameter [FL] along the orthogonal, $FO = FU$, so angle FOU = angle FUO.⁶⁹ However, since the [remaining] two angles POU and OPU [of triangle POU] sum up to a right angle, angle FUP = angle FPU [because triangle OFU = triangle PFU, since

OU and OP are both bisected by FL, and OP is bisected by FU], so $FP = FU$, and so it equals FO. And therefore $PO = BG$ [because FP was constructed to be half of BG, and it forms half of PO], and it is also equal to GD [since GD and BG are both radii], so $EC:GD = TQ:PO$ [since $TQ = EC$, by construction].

[2.189] But since right angle $KDG =$ angle GID [which is right by construction], while angle IGD is common, triangle IGD will be similar to triangle KGD [by Euclid, VI.4], and $GD:DI = GK:KD$. Yet angle $KGD =$ angle OPU [by construction], whereas right angle $KDG =$ [right angle] OUP , so triangle KDG is similar to triangle OUP [by Euclid, VI.4], and $KG:KD = OP:OU$. Hence, $DG:DI = OP:OU$. Accordingly, $EC:DI = QT:OU$.⁷⁰

[2.190] But $QT:OU = TF:FU$, since triangle TFQ is similar to triangle OFU [and QT and OU and TF and FU are corresponding sides]. However, angle $UTS =$ angle HFU , since it is alternate to it [because PQ and ST are parallel, by construction], and right angle $UST =$ [right] angle FHU [both being right by construction. Consequently] triangle UST will be similar to triangle HUF , so $TU:UF = SU:UH$, and so [by Euclid, V.18] $TF:UF = SH:UH$. But $TN = SH$, since it is parallel to it and since both lie between two parallels [PQ and ST]. Therefore, $TF:UF = TN:UH$, so $QT:OU$ [which = $TF:UF$, by previous conclusions] = $TN:UH$, and $EC:DI$ [which = $QT:OU$, by previous conclusions] = $TN:UH$.

[2.191] But since right angle $GID =$ [right] angle PHU , and since angle $IGD =$ angle HPU [by construction], triangle IGD will be similar to triangle HPU [by Euclid, VI.4], and $ID:GD = HU:UP$, so $EC:GD = TN:UP$.⁷¹ Yet, since angle $CGE =$ angle NPT [by construction], and since right angle $GCE =$ [right angle] PNT , [then triangles CGE and NPT are similar, so] $GE:EC = PT:NT$. Hence, $GE:GD = PT:UP$.⁷²

[2.192] But angle $DGE =$ angle UPT [because angle QPT was constructed equal to angle EGB , and angle $DGB =$ angle HPU , so the remainders DGE and UPT are equal]. Therefore, triangle DGE is similar to triangle UPT . Accordingly, angle $GDE =$ angle PUT . It therefore follows that angle GDZ [of triangle GDZ] = angle PUF [of triangle PUF], and angle $DGZ =$ angle UPF , so the third [angle DZG] = the third [angle PFU , so the triangles are similar], and $DZ:ZG = UF:FP$. But $UF = FP$ [by previous conclusions]. Therefore, $DZ = ZG$, which is what was proposed.

[2.193] **[PROPOSITION 24, LEMMA 6]** Furthermore, given a right triangle ABG [figure 5.2.24, p. 238] with right angle ABG , and given point D on either BG or AB , to draw a line from point D to side AG intersecting it at point Q [on one side] and intersecting the other side [at point T] in the other direction such that the sum [of the lengths from D to the respective sides] is to GQ as E is to Z [i.e., $(TD + DQ = TQ):GQ = E:Z$].

[2.194] For instance, from point D draw DM parallel to AB, and produce a circle passing through the three points D, M, and G. MG will be a diameter [since angle MDG is right, by construction]. Draw line AD, and [by Euclid, VI.12] let H be a line in proportion to which AD is as E is to Z [i.e., $AD:H = E:Z$]. Since angle DMG = angle BAG, cut from it [an angle] equal to angle DAG, and let it be CMD. Draw MC until it meets the circle at point C, and from that point draw a line to diameter MG and [extend it from point of intersection L] to the circle so that LN = line H [by proposition 20, lemma 2]. Then draw line NG, and [draw] line DN to intersect AG at point Q.

[2.195] Accordingly, since angle DMC = angle DNC, because they are subtended by the same arc, and since angle QNL = angle DAQ [because $QNL = DMC$, which = DAG, by construction], and since angle NQL = [vertical] angle DQA, then [by Euclid, VI.4] triangle NQL is similar to triangle DQA. Therefore, $AQ:QN = AD:NL$.

[2.196] But, since angle DMG = angle DNG [because they are subtended by the same arc DCG, then angle] QNG = angle TAQ [because $TAQ = DMG$, which = DNG]. Let T be the point where DN intersects AB, and [since] angle TQA = [vertical] angle NQG, [then, by Euclid, VI.4] triangle TQA will be similar to triangle NQG, and $AQ:QN = TQ:QG$. Therefore, $TQ:QG = AD:LN$ [since $AQ:QN = AD:LN$, by previous conclusions]. But $NL = H$, and $AD:H = E:Z$ [by construction]. Therefore, $TQ:QG = E:Z$, which is what was proposed.

[2.197] Furthermore [as demonstrated in proposition 20, lemma 2], it is possible for two lines like CN to be drawn, and in that case two lines can be drawn from D equal to TQ such that each of them is to the segment it cuts off from AG as E is to Z, and the proof will be identical.⁷³

[2.198] **[PROPOSITION 25]** With these things established, and given a [convex] spherical mirror, it will be [shown how] to find a point of reflection on it.

[2.199] For instance, let A [figure 5.2.25, p. 239] be a center of sight, B a visible point, and G the center of the sphere [forming the mirror], and draw lines AG and BG. Take the plane within which these two lines lie, and take the [great] circle [that forms the] common [section] of this plane and the mirror. Accordingly, the point of reflection will be found on this circle.

[2.200] Take some other line MK, and divide it at point F such that $FM:FK = BG:GA$ [by Euclid, VI.10]. Bisect MK at point O, and from point O draw perpendicular CO, and from point K draw line KC to CO to form an angle [OCK] with CO that is equal to half of angle BGA. Then, from point F draw line FP to CK, and let it intersect CO at point S so that $SP:PK = BG:\text{radius } GD$ [by proposition 24, lemma 6]. From angle BGA cut off an angle equal to angle SPK, i.e., [angle] DGB, and draw lines SK and BD.

[2.201] Accordingly, $BG:GD = SP:PK$ [by construction], and so triangle SPK will be similar to triangle BGD [by Euclid, VI.6, because angle $SPK =$ angle DGB , by construction], and [therefore] angle $SKP =$ angle BDG . But, according to what we established earlier [in proposition 24, lemma 6], we may draw from point F to CK another line like SP [i.e., $S'P'$] such that it is to the segment it will cut off from CK as SP is to PK [i.e., $S'P':P'K = SP:PK$], and on that basis a line other than SK [i.e., $S'K$] will be drawn from point K to OS to form another angle $[CKS']$ with CK [i.e., other than the original CKS] that is either greater than, or less than angle CKS . If the larger of these angles is not greater than a right angle, no point of reflection will be found [as will be demonstrated below]. Accordingly, let angle CKS be greater than a right angle, and the point [of reflection] is found as follows.

[2.202] Angle BDG will be greater than a right angle [since $BDG = CKS$, by previous conclusions, and CKS is greater than a right angle, by stipulation]. Draw tangent NDY , and since angle PKO is smaller than a right angle, cut off angle QDG equal to it from angle BDG . Thus, since angle $SPK =$ angle QGD [i.e., DGB , by construction], triangle FPK will be similar to triangle QGD [by Euclid, VI.4, so angle $DQG =$ angle KFP , and so] angle DQB [adjacent to DQG] = angle KFS [adjacent to KFP], and triangle DQB will be similar to triangle KFS [since angle $SKP =$ angle BDG , and angle $QDG =$ angle PKF , leaving remaining angles QBD and FKS equal].

[2.203] Now, extend DQ [beyond Q], and from point B draw perpendicular BZ to it. Accordingly, angle $BQZ =$ angle SFO [since triangles BQD and SKF are similar and BZ and SO are dropped orthogonally to corresponding sides from the corresponding vertices]. Also, right angle $BZQ =$ [right] angle SOF , and so triangle BQZ is similar to triangle SFO [by Euclid, VI.4].

[2.204] Extend DZ to point I , and let $ZI = ZD$. It is therefore evident that $ZQ:QB$ and $QB:QD$ are as $OF:FS$ and $FS:FK$, and from this $ZD:QD = OK:FK$ [by Euclid, V.22], so [by Euclid, V.18] $ID:QD = MK:FK$ [since ID and MK are, by construction, twice ZD and FK , respectively], and so [again, by Euclid, V.17] $IQ:QD = MF:FK$, and $IQ:QD = BG:GA$ [because $MF:FK = BG:GA$, by construction].

[2.205] Now, draw line BI and [draw line] DL parallel to it. Triangle LDQ will be similar to triangle BQI [by Euclid, VI.4, since angle $QBI =$ alternate angle DLQ , angle $BIQ =$ alternate angle LDQ , and angle $BQI =$ vertical angle DQL], and [so] $IQ:QD = IB:DL$. And since $IZ = ZD$ [by construction], and since BZ is perpendicular [to ID], $BD = BI$, so $BD:DL = BG:GA$ [because $IQ:QD = BG:GA = IB:DL = BD:DL$, by previous conclusions].

[2.206] Now, from point D draw line DH to form an angle $[HDL]$ with line LD that is equal to angle BGA . Then, since HL and DL intersect, the two angles LHD and LDH will sum up to less than two right angles, and so

the two angles AGH [which = LDH, by construction] and DHG that are equal to these sum up to less than two right angles, so HD will intersect GA. I say that it will intersect at point A.

[2.207] It is evident that right [angle] GDN = the [sum of the] two angles OCK and OKC [in triangle OKC where KOC is a right angle, by construction, and angle NDG is also right, by construction], and angle OKC = angle GDQ [by construction]. It follows that angle QDN = angle OCK [because angles QDN and QDG sum up to a right angle, as do OCK and OKC, and angle QDG = angle OKC in similar triangles QDG and FPK], so angle QDN = half of angle BGA as well as half of angle HDL [since angle QDN = angle OCK = half of angle BGA, by construction, and angle HDL = angle BGA, by construction]. But angle QDB is half of angle BDL, because BQ:QL = BD:DL, since triangle DLQ is similar to triangle BQI [by previous conclusions], and BD = BI [by previous conclusions]. It therefore follows that angle NDB is half of angle HDB, and so [angle] BDN = [angle] NDH. It follows, moreover, that [when radius GD is extended beyond the circle to E, angle] BDE = angle HDG [because NDE and NDG are both right angles, and NDH = NDB, so NDE – NDB = BDE = NDG – NDH = HDG]. But angle HDG = vertical angle EDA, so [angle] BDE = [angle] EDA, and so D is the point of reflection. I say this is so if HD intersects AG at point A, which will be demonstrated as follows.

[2.208] Draw line HT parallel to BD. It is clear that angle BDE = angle HDG [by previous conclusions]. But [angle] BDE = [alternate] angle HTD, so [triangle HDT is isosceles, and so] HT = HD. But [given that HT is parallel to base BD of triangle BDG, triangles BDG and HTG are similar, so], BD:HT = BG:GH, as Euclid demonstrates [in VI.2]. Thus, BD:DH [which = HT] = BG:GH. But, when it is extended, HD will intersect GA, and it will form a triangle [HGA] similar to triangle HDL [by Euclid, VI.4], since it has angle LHD in common [with triangle HDL], and since angle HDL = angle HGA [by construction]. Therefore, HD is to DL as HG is to the line [X] that HD cuts from GA [i.e., HD:DL = HG:X]. But BD:DL is compounded from BD:DH and DH:DL [i.e., BD:DL = (BD:DH):(DH:DL)]. It is therefore compounded from BG:GH and GH:[X, i.e.,] the line HD cuts off from GA [i.e., BD:DL = (BG:GH):(GH:X)].⁷⁴ But BD:DL = BG:GA [by construction]. Therefore, BG:GA is compounded from BG:GH and GH:[X, i.e.,] the line HD cuts from GA [i.e., BG:GA = (BG:GH):(GH:X)]. But it is compounded from BG:GH and GH:GA [so GA = X]. Therefore, GA is the line HD cuts off from GA, and so it will intersect it at point A, which is what was proposed.

[2.209] If, however, angle CKS is not greater than a right angle, I say that reflection will not occur to the center of sight from any point on the mirror.

[2.210] For if it is claimed that [such reflection] can [occur], let D be the point of reflection, and draw line AD to point H on diameter BG. Make

angle LDH equal to angle AGB, draw tangent NDY, and make angle QDN equal to half of angle AGB.

[2.211] It is evident [from previous conclusions] that triangle HDL is similar to triangle HGA, so $DH:DL = HG:GA$. But $BD:DH = BG:GH$, which will be evident from the fact that HT is parallel to BD [by construction]. So $BD:DL = BG:GA$. However, since angle BDE = [alternate] angle HDG, angle BDN will be half of angle BDH [by previous conclusions]. But [angle] NDQ is half of angle HDL [by previous conclusions]. Therefore, [angle] BDQ is half of angle BDL [by previous conclusions], so $BQ:QL = BD:DL$ [by Euclid, VI.3].

[2.212] From point B draw BI parallel to DL, and let DQ intersect it at point I. Bisect DI at point Z, and draw BZ. Triangle BQI will be similar to triangle QDL [by Euclid, VI.4]. Therefore, $BQ:QL = BI:DL$, and so $BI = BD$ [by previous conclusions]. But $IQ:QD = MF:FK$ [by previous conclusions], so $ID:QD = MK:FK$ [by Euclid, V.18], $DZ:QD = OK:FK$ [by Euclid, V.17, since $OK = \text{half } MK$, and $IZ = \text{half } ID$], and $ZQ:QD = OF:FK$ [by Euclid, V.17].

[2.213] It is evident that BZ is perpendicular [to IQ]. Extend it until it intersects DG at point X, which is possible, because angle DZX is a right angle, while ZDX is less than a right angle. And it is obvious that $BG:GD = SP:PK$ [by construction]. Therefore, since it is claimed that angle CKS is not greater than a right angle, I say that an angle greater than a right angle will be formed at point K by a line that intersects CO at the point from which a line passing to CK through point F is to the segment of CK [cut off by it] as $BG:GD$ [i.e., $SP:PK = BG:GD$].⁷⁵

[2.214] For example, it is clear that, since angle QDN = angle KCO [by previous conclusions], angle QDG = angle CKO. Accordingly, at point K form an angle equal to BDQ [i.e., SKO], and suppose that the line forming this angle intersects CO at point S, and draw SFP. It is evident that, since right angle BZD = [right] angle SOK, triangle BZD will be similar to [triangle] SOK, and $BZ:BD = OS:SK$. But $QZ:QD = OF:FK$ [by previous conclusions]. Hence, angle ZBQ = angle OSF, and angle QBD = angle FSK, so triangle BGD is similar to triangle SPK. Therefore, $SP:PK = BG:GD$, which is what was proposed.⁷⁶

[2.215] Furthermore, it is impossible for two angles to be erected on MO such that both of them are larger than a right angle. For if both such angles were larger than a right angle, then, since an angle equal to angle SKM may be formed upon the same center, another angle different from this one will be formed upon the same center which will form on KM another line like SK. And so reflection will occur from point D as well as from some other point on this circle, which is impossible, since it has already been demon-

strated [in proposition 16 above] that for any one center of sight there is [only] one point of reflection, and it has now been shown how to find it.⁷⁷

[2.216] Given two eyes, even though there are two points of reflection, there will nonetheless be a single image according to sense-deduction, and there will be a single image-location.⁷⁸ We will demonstrate this on the assumption that the two lines extending from the centers of the eyes to the center of the circle are equal.

[2.217] Now, if the location of the visible point is the same with respect to both eyes, so that the lines [extending] from the visible point to the centers of the eyes are equal, the proof will be simple, because the visual axes cut an arc of reflection on the circle, and they form equal angles with the line extending from the visible point to the center of the sphere [i.e., the normal], and the arcs lying between this line and the visual axes are equal. And if the points [of reflection] are selected according to the previous proof [in proposition 25 above], the arcs on the circle lying between these two points and the point on the circle that lies on the normal extended from the visible point [to the circle's center] will be equal, which will be easily demonstrated by repeating the preceding demonstration.

[2.218] And this is the case whether the points of reflection lie in the same plane of reflection, or in different ones; those arcs will still be equal, the lines extending from the centers of the eyes to the points of reflection will be equal, and the lines [extending] from the visible point to the same points will be equal. In addition, the lines extending from the centers of the eyes to the points of reflection will necessarily intersect one another, and the proof is obvious that the intersection will be at the same point on the normal dropped from the visible point, and at this point one and the same image will appear to both eyes, which is what was proposed.⁷⁹

[2.219] Furthermore, the arrangement of images is the same as the arrangement of the visible points [producing them]. For if a line [forming a cross-section] is taken on a visible object, and if two lines are drawn from its endpoints to the center of the sphere, it will form a triangle within which the images of all the points on that [cross-sectional] line will be included. And if there is a point on that line that is identically disposed [with respect to both eyes], the image of a point farther from it will lie on a normal farther from its normal, and one nearer [will appear] on a nearer [normal]. And so, in images [any] portion maintains its [relative] position as it actually exists among the visible points [within that portion].

[2.220] Moreover, given a line on which there is [such] a uniformly disposed point, any point on that line will be uniformly disposed with respect to both eyes, according to the previously discussed way, and it will have a single image because of the equality of angles formed by that line with the

visual axes. In addition, if a line is chosen that bisects the angle formed by the two lines from the centers of the eye to the visible point, the location of any point on that line, no matter how far it is extended, will be the same for both eyes, just as was the case for the other line, and the same method of demonstration applies.

[2.221] Aside from these two lines, no other one can be found that maintains the same location, so, when a visible point is perceived on the normal, its image will fall at different points on that normal, but [the two resulting images will lie] an imperceptible distance from each other. Also, the image of any point, no matter how many eyes may view it, always maintains a uniform relative position, so the image appears unified, as has been claimed in the case of direct vision. For, even though they may fall at different places [on the normal], still, because of their insensible distance [from one another], the images do not appear disparate unless their relative positions are disparate. In a similar vein, when the distance of a point from one eye is slightly greater than it is from the other, the image-locations will [only] be imperceptibly separated, so they appear fused, and one [image] is melded from them, in which case the image-locations may be partially rather than wholly distinct.⁸⁰

[Convex Cylindrical Mirrors]

[2.222] In convex cylindrical mirrors, the common section of the plane of reflection and the mirror's surface is sometimes a straight line, sometimes a circle, and sometimes a cylindric section [i.e., an ellipse].⁸¹

[2.223] When the common section is a straight line, the image-location will lie on the normal extending from the visible point to the surface of the mirror, and that image-location lies just as far [below the mirror] from the common section [i.e., the line of longitude] as the visible point lies [above it]. [This is subject to] the same proof as applies to the plane mirror [in proposition 2 above].

[2.224] However, when the common section is a circle, the image-location will sometimes lie inside the circle, sometimes outside it, and sometimes on the circle itself. This phenomenon has the same explanation as applies in the convex spherical mirror [in propositions 11-15 above].

[2.225] But if the common section is a cylindric section, I say that some of the image-locations lie inside the mirror, some on the mirror's surface, and some outside the mirror. These claims will be explained individually.

[2.226] **[PROPOSITION 26]** Let ABG [figure 5.2.26, p. 241] be a cylindric section, B the point of reflection, E the visible point, and D the center of

sight. From point B draw a normal to the plane tangent to the mirror at point B, let it be TBQ, and from point E draw perpendicular EKQ to the plane tangent to the mirror at point K [to form the normal dropped from the object-point]. Let the line tangent to the mirror at point B be CU; let the line tangent to the mirror at point K be KM. I say that the two normals TB and EQ will intersect.

[2.227] Draw lines EB and DB, and draw line KB. It is obvious that KM will fall within figure EKB, and line BC [will fall] within the same figure. Therefore, BC will intersect EK. Let it intersect at point C. It is evident that angle TBK > right angle [CBT], and angle EKB is likewise greater than right angle [EKM], so TB and EK will intersect [since angles BKQ and KBQ, adjacent to angles EKB and TBK are acute]. Let Q be the point of intersection. Likewise, [angle] DBK is greater than a right angle. Therefore, DB and EK will intersect. Let H be the point of intersection. Accordingly, H is the image-location. I say, as well, that $EQ:QH = EC:CH$, and also that $QH > HB$.

[2.228] Draw HF parallel to EB. It is clear that angle EBC = angle DBU [because angle of incidence EBT = angle of reflection DBT by construction]. It is therefore equal to angle CBH [which is vertical to angle DBU]. It follows that [angle] EBT = angle HBQ, since TBC is a right angle, and QBC is a right angle [and angle EBT = TBC – EBC, whereas angle HBQ = CBQ – CBH, and EBC = CBH, so the remainders EBT and HBQ are equal]. Thus, since CB bisects angle EBH, $EC:CH = EB:BH$ [by Euclid, VI.3].

[2.229] But angle EBT = [alternate] angle HFB [since HF and EB are parallel, by construction], so HF and HB are equal [insofar as they form equal angles with FB]. But $EB:HF = EQ:QH$ [because, by Euclid, VI.4, triangles EBQ and HQF are similar, since angle QHF = alternate angle QEB, and angle HFQ = alternate angle EBQ]. Accordingly [since $EC:CH = EB:BH = EB:HF = EQ:QH$, by previous conclusions], $EC:CH = EQ:QH$, which is what was proposed.⁸² On this basis, moreover, because $EQ:QH = EB:BH$, and because $EQ > EB$, $QH > HB$, which is [also] what was proposed.

[2.230] From this it is evident that, if a perpendicular is dropped to a plane tangent to segment GB [of the mirror], it will intersect TB. By the same token, any [perpendicular] dropped to segment AB [of the mirror] will intersect TB. And these conclusions are evident when the visible point does not lie on the visual axis. For [when it does] it is clear from earlier discussions [in proposition 8 above] that the form of one single point reaches the mirror orthogonally and is reflected back along the same line, and this point [whose form reaches the mirror along the] normal lies on the surface of the eye, for a point taken outside the eye cannot be reflected along this normal because it cannot reach the mirror along this normal according to the aforementioned reasoning [i.e., that the body of the eye gets in the way].

By the same token, it could not be reflected from a point on the mirror other than from a point on the normal, because [in that case] two normals would happen to intersect and form a triangle with two right angles, just as was shown above [in proposition 8].

[2.231] **[PROPOSITION 27]** Furthermore, take [some] cylindric section [figure 5.2.27, p. 242], select point A on it, draw tangent [E]AT to the section [at point A], and within the [section on the] mirror take DA perpendicular to [E]AT.

[2.232] It is obvious that AD divides the [cylindric] section into two portions, in either of which there is a single point to which the tangent will be parallel to AD. Accordingly, let G [be a point in one of the two portions of the section] to which the tangent intersects AD at point H, and to this tangent draw perpendicular QG, which will necessarily intersect HD, as was shown in the preceding figure [i.e., 5.2.26]. Let D be the point of intersection, draw line GA to P, and draw line QA. Accordingly, angle QAH is equal to, greater than, or smaller than angle HAP.

[2.233] Let it be equal. The form of point Q will therefore reach A and will be reflected to P, which is the center of sight, and the image-location will be a point on the cylindric section, i.e., G [because G is where line of reflection PA intersects normal QD].

[2.334] If, however, some point, such as point F, is taken above point Q, angle FAH < angle HAP. Make it equal to [angle] NAH. [Line of reflection] NA will intersect [normal] GQ inside the cylinder. Let it [do so] at point K. It is evident, then, that the image of point F will lie at point K, and the images of all points beyond Q [will lie] inside the cylinder.

[2.235] But if some point, such as C, is taken between Q and T, angle CAH > angle HAP. Make it equal to HAM. It is clear that [line of reflection] MA will fall on [normal] GQ, but outside the [cylindric] section. Let it [do so] at point O. Thus, the image of point C will lie at point O, and the images of all points lying between T and Q will lie outside the [cylindric] section between T and G.

[2.236] Moreover, if angle QAH < angle HAP, then cut [an angle] from HAP equal to it, and let it be HAN. It is evident that the image of Q will lie at point K, and the images of all points above [Q] will lie within the [cylindric] section. But if point C is taken below [Q] so that angle CAH = angle HAP, the image of C will lie on the [cylindric] section, [the images of] all [points] between C and Q [will lie] inside [the cylinder], and [the images of] all [points] between C and T [will lie] outside [the cylinder].

[2.237] If, however, angle QAH > angle HAP, make [angle] HAM [figure 5.2.27a, p. 242] equal to it. It is clear that MA will intersect the [cylindric]

section [at some point beyond A on arc AG]. Let it intersect at point B, draw the tangent to point B, and let it intersect DH at point L. Now, angle DLB will be acute, angle HLB will be obtuse, and, when it intersects HG, LB will form an acute [angle] with it. Draw SB perpendicular to LB at point B. It will intersect HG, it will form an acute angle with it, and its vertical angle will likewise be acute. Let HG intersect QA. Let U be the point of intersection, and it forms an acute angle with QA at point U [i.e., angle HUA is acute], so SB and QU intersect. Let the intersection be at point Z. It is therefore evident that the form of point Z will reach the mirror along ZA, and it is reflected along AM [since QAH = HAM, by construction], and B [will be] the image-location. Moreover, the images of points above Z on line ZS will lie inside the [cylindric] section [because both the new ZAH and MAH will be more acute], whereas [the images] of points below Z [on line ZS] will lie outside the [cylindric] section [because both the new ZAH and the new MAH will be more obtuse], which was what was proposed.⁸³

[2.238] **[PROPOSITION 28]** Moreover, reflection occurs to a center of sight from only one point on a cylindrical mirror, as, for example, [the form of] point B [figure 5.2.28, p. 243] is reflected to [point] A from point G. I say that it does not reflect to the same point from any point on the mirror other than from point G.

[2.239] For if the entire axis [CED] of the mirror lies in the plane of reflection ABG, the common section of the mirror's surface and the plane of reflection will be a line of longitude [FGN] on the mirror. And since the center of sight [A], the visible point [B], the point of reflection [G], and the point [E] on the axis where the normal [to the point of reflection] falls [all] lie in the [same] plane of reflection, only one plane can be assumed within which that line of longitude, or the axis, and points A and B lie, so reflection can only occur to A from some point on the line of longitude [FN]. But it has already been demonstrated [in proposition 3] that reflection cannot occur to [point] A from any point other than G on the line of longitude, so in this case reflection occurs to A from only one point on the mirror.⁸⁴

[2.240] But if the plane [of reflection] A'B'G is parallel to the base of the cylinder, the common section [of this plane with the cylinder] will be a circle [GH] parallel to the base. And it has already been demonstrated [in proposition 16] that reflection to [point] A' cannot occur from any other point on that circle. But if reflection were to occur from some other point on the mirror [outside circle GH], the normal dropped from that point would fall orthogonally to the axis [CED], and it would intersect line A'B' at some point [K]. From that point draw a line [KLI] to the axis in the plane parallel to the base of the cylinder. It will of course be orthogonal to the axis, and so

two perpendiculars [KE and KI] will form with the axis a triangle [KEI] two of whose angles [KEI and KIE] are right angles, which is impossible. It is therefore evident that in this case B does not reflect to A except from point G.⁸⁵

[2.241] If, however, plane ABG cuts the mirror according to a cylindric section, I say that reflection occurs from point G only.

[2.242] From [the center of sight at] point A [figure 5.2.28a, p. 243] produce a plane parallel to the base of the cylinder, let it be EZI, and likewise from point [of reflection] G produce a plane parallel to the base of the mirror [i.e., GSP] and in that plane draw line TG from the [mirror's] axis [TQ] to point G. This line will therefore be perpendicular to the plane tangent to the mirror at point G. Let it intersect [line] AB at point K, and from point G draw GZ, the line of longitude on the mirror, and let TQ be the axis. Then, from [object-]point B draw BH perpendicular to plane EZI, and draw lines AZ and HZ. In that same plane [i.e., EZI] draw line ZQ from Z to the axis. It will be perpendicular to the axis, since the axis is perpendicular to this plane [EZI within which it lies], and it will be perpendicular to the plane tangent to the mirror at point Z. Let it also intersect line AH at point L. I say that the form of point H is reflected to A from point Z.

[2.243] From point A draw [line] AM parallel to line KG, and it will intersect BG. Let M be the point of intersection. It is evident that [line] GZ is parallel to line BH, since [by construction] both of them are perpendicular to parallel planes [i.e., EZI and GSP], so line BGM lies in the same plane [BGMH] as these lines [GZ and BH]. Accordingly, the three points M, Z, and H lie in this plane. But AM in turn is parallel to KG, and LZ is parallel to KG, because GZ is parallel to TQ and lies between parallel planes [i.e., EZI and GSP]. Therefore, LZ is parallel to AM, so they lie in the same plane [AHZM], and line AH lies in it too. Hence, the three points M, Z, and H lie in this [same] plane. But it has already been shown that they lie in plane M[G]BH. So these two planes [AHZM and MGBH] intersect along a common line [MZH]. Therefore, HZM is a straight line.

[2.244] Since G is the point of reflection, it is evident that angle [of reflection] AGK = angle [of incidence] KGB, and so angle AGK = angle AMG [which is alternate to angle KGB between parallels AM and KG]. But it is equal to MAG, because it is alternate to it [between the same two parallels]. Hence, [within isosceles triangle MAG] AG and MG are equal. But since GZ is orthogonal to any line within plane AZH, [then, by the Pythagorean Theorem] $MG^2 = MZ^2 + GZ^2$ [in right triangle MZG]. By the same token, $AG^2 = AZ^2 + GZ^2$ [in right triangle AZG]. Hence, $AZ = MZ$ [because $AG = MG$, by previous conclusions], so angle AMZ = angle ZAM. But [since AM and LZ are parallel, by previous conclusions] angle AMZ is also equal to

[alternate] angle LZH, and angle ZAM = angle LZA, since they are alternate angles. Therefore, angle AZL = angle LZH, so, when it reaches point Z, the form of point H is reflected to point A.

[2.245] Hence, if it is claimed that the form of B can be reflected to A from some point [D] other than G, that other point will lie either on line of longitude GZ or on some other [line of longitude]. If it lies on GZ, draw from it a perpendicular [DL'], which will necessarily intersect line AK and will be parallel to line AM [because, by supposition, it must lie in a plane of reflection that includes points A and B, and it must be normal to the mirror]. In addition, the line extending from point B to that point [D] will necessarily intersect line AM [because line AM is in the plane of reflection that includes A and B], and [so] that point and point M will lie in the same plane.

[2.246] Furthermore, that line [BD] will either fall on point M or [will fall] on some other point. If [it falls] on point M, then two [different] straight lines will have been drawn from point B to point M [i.e., BGM and BDM]. On the other hand, if [it falls] to another point on line AM [i.e., N], draw a line from that point to point Z, and it is demonstrated that this line forms a straight line with HZ, as was demonstrated for line ZM. And so two straight lines will have been drawn from point H to pass through point Z and fall at different points on AM, which is impossible.

[2.247] It is therefore evident that [the form of] B can be reflected to A from no point other than G on line GZ. If it is claimed that [it can be so reflected] from a point taken outside this line, draw the line of longitude on the mirror that [lies] on this point, and from the point on circle EZI where this line [of longitude] falls, it is demonstrated that [the form of] H is reflected to A according to the previous proof. But it has already been demonstrated that [the form of] H is reflected to A from point Z, so [the foregoing conclusion] is impossible [by proposition 16 above]. It therefore follows that [the form of] B is reflected to A from only one point on the mirror, which is what was proposed.

[2.248] **[PROPOSITION 29]** Furthermore, given that [the form of] point B is reflected to A, it will be possible to find the point of reflection, and this will be demonstrated by reversing the [previous] proof.

[2.249] From point A [figure 5.2.28a, p. 243] produce a plane parallel to the cylinder's base, and this plane will cut the cylinder along circle EZI. Draw line BH orthogonal to this plane from point B, and find point Z within this plane from which the reflection of [the form of point] H occurs to A [by proposition 25 above]. From point Z draw line of longitude ZG, and from point Z [draw] perpendicular ZL, to which AM [extended] from point A is parallel. Extend line HZ until it intersects the latter, and let M be the point

of intersection. From point M draw a [straight] line to B, and it will necessarily intersect line ZG, since it lies in the same plane with it. Consequently, since BH is parallel to GZ, HZM will lie in the same plane with them, and so MB [will lie] in that same plane, and if it intersects ZG at point G, then point G will be the point of reflection, and you can see this if you reverse the previous demonstration.⁸⁶

[Convex Conical Mirrors]

[2.250] In convex conical mirrors, if the common section of the plane of reflection and the surface of the mirror is a line of longitude on the mirror, the image-location will be just as it was determined in the case of plane mirrors, and the same proof [applies].

[2.251] That the common section cannot be a circle is evident from the fact that the normal [must] fall orthogonally to the plane tangent to the mirror at the point of reflection, and [the plane of] the circle will necessarily be parallel to the [mirror's] base. But no [plane] parallel to [the plane of] the base will be orthogonal to a plane tangent to the mirror.

[2.252] If, however, the common section is a conic section, some images will lie on the surface of the mirror, some inside the mirror, and some outside it. And [all this] is determined by the same method as [was applied] in the case of the convex cylindrical mirror, and the same proof [applies]. Furthermore, just as was shown in the case of the convex cylindrical [mirror], the form of only one point on the surface of the eye is reflected orthogonally along the visual axis to the eye, and it does so from only one point on the mirror, and its image-location is continuous with the other image-locations [surrounding it], as was shown above.

[2.253] It remains [for us] to show that, in these sorts of mirror [i.e., conical convex], reflection may occur from only one point on the mirror, which we will prove as follows.

[2.254] **[PROPOSITION 30]** Let A [figure 5.2.30, p. 244] be the center of sight, B the visible point, G the point of reflection, and on point G produce a plane parallel to the base, a plane that will cut the cone along circle PG. Draw lines AG, BG, and AB, and from point G draw line GT to the center of the circle. Let the vertex [of the cone] be E, and from it draw axis ET. Then [in the plane of ETG] draw perpendicular HG to the plane [containing line CG that is] tangent to the mirror at point G, and, since [normal] HG bisects angle AGB, it will fall upon AB. Let Z be the point where it falls [on AB].

[2.255] From the vertex draw line of longitude EG on the mirror to point G, and draw a line from point A parallel to this line, and this parallel will

necessarily intersect the plane of circle GP. Let it be AN, and let it intersect at point N. Likewise, from point B draw a parallel to EG, i.e., BM, and let it intersect the plane [of circle] PG at point M. From point N, as well, draw NF parallel to GT, and draw lines NG, MG, and NM.

[2.256] It is evident that TG will intersect NM. Let it intersect at point Q. It is also evident that MG will intersect NF, since it intersects [line TG] parallel to it. Let F be the point of intersection. Then, from point A draw AL parallel to HZ. It is clear that BG will intersect AL [since A, B, G, and Z lie in the same plane, and AL is parallel to HZ, by construction]. Let L be the [point of] intersection. Then draw GC, which is the common section of the plane [CGE] tangent to the mirror at point G and the plane of circle PG. It is clear that it will be orthogonal to GT, as well as to NF [because NF was constructed parallel to GT].

[2.257] In addition, take GD, which is the common section of the plane tangent [to the mirror at point G] and the plane of reflection [AGBZ], and GD will intersect AL, since it intersects GH [to which AL was constructed parallel]. Let D be the point of intersection, and GD will be perpendicular to AL [since it is perpendicular to ZG, which is perpendicular to plane CGED tangent to the mirror at point G and parallel to AL].

[2.258] It is clear from the foregoing that NF is parallel to GT [by construction], and AL is parallel to GH [by construction]. Therefore, the plane containing NF and AL [i.e., FLAN] is parallel to plane G[E]TH. But EG is parallel to BM, so they lie in the same plane, and that plane intersects the aforementioned parallel [planes, i.e., GETH and FLAN], one along line [of longitude] EG, the other along line FL, so FL is parallel to EG. But AN is parallel to that same line [EG, by construction]. Therefore, FL is parallel to AN.

[2.259] However, the plane tangent to the mirror at point G intersects the same parallel planes [i.e., GETH and FLAN], one along line [of longitude] EG, the other along line CD. Therefore, CD is parallel to EG. It is therefore parallel to AN and LE, so $AD:DL = NC:CF$ [by Euclid, VI.1, VI.2, and V.11].

[2.260] It is evident as well that angle [of incidence] BGZ = angle [of reflection] ZGA [by construction], and it also equals [alternate] angle GLA [between parallels HZ and AL], as well as angle GAL [which is alternate to angle AGZ between the same parallels], so [angles] GAL and GLA are equal. In addition [because triangle GAL is isosceles], GA and GL are equal, and GD is perpendicular to AL [by previous conclusions. So] $AD = DL$. Therefore, $NC = CF$ [since $AD:DL = NC:CF$, as previously concluded], and GC is perpendicular [to FN. So] angle CFG = angle CNG. Hence, angle NGQ = angle MGQ.⁸⁷ Accordingly, [the form of] point M can be reflected to [point] N from point G on circle PG, barring interference from the cone [itself].⁸⁸

[2.261] I say, then, that [the form of] point B is reflected to A only from G. For if it is claimed that it can be reflected from some other point, that point will either lie on line of longitude EG, or it will not.

[2.262] Let it lie on it, let it be X [figure 5.2.30a, p. 245], and from that point draw the normal to the plane tangent to the mirror at that point, and this normal will be parallel to ZG and therefore parallel to AL. AL will therefore lie in the plane of reflection containing this normal, and it will likewise lie in the plane of reflection containing normal ZG. Hence, those two planes of reflection intersect along line AL. But they intersect on point B, which is impossible, because B does not lie on line AL, which is evident from the fact that FL is parallel to BM.⁸⁹ It therefore follows that [the form of point] B can be reflected to A from no other point on line EG than G.

[2.263] But if [it were assumed to do so] from some other point [not on line of longitude EG], then let that point be U, draw line of longitude EUO, and take the plane parallel to base [PG of the mirror] passing through point U. It is clear that AN will intersect this plane. Let Y be the point of intersection. BM will intersect the same plane as well. Let K be the point of intersection, and draw lines KU, YU, and YK. Since that plane intersects the cone along a circle passing through U, draw line RU from point U to the center of this circle. Draw lines EK and EY, which [when extended] will intersect the plane of circle PG, and let I and S be the points of intersection. Now, draw lines IO and SO.

[2.264] Thus, just as it has been demonstrated [earlier in this proposition] for point M that, barring interference from the cone, it[s form] can be reflected to N from point G, so it is demonstrable for point K that it[s form] can be reflected to point Y from point U, and the proof is the same. And so angle RUY = angle RUK.

[2.265] It is therefore evident that BK is parallel to EG [by construction], and the common section of plane BGEK and the plane of circle PG is line MG. Therefore, since it lies in that plane [i.e., BGEK] and intersects the plane of circle PG, line EK will fall on common section MG. SMG will therefore be a straight line.

[2.266] By the same token, since plane NYEG intersects the plane of circle PG along line NG, line EY will intersect line NG [at point I]. Thus, ING is a straight line. It is also evident that plane IOE intersects the plane of circle PG along line IO, and it intersects the plane parallel to this one that passes through U along line YU. Therefore, YU is parallel to IO. Likewise, plane SOEK intersects those parallel planes along the two lines SO and KU. Therefore, SO is parallel to KU.

[2.267] Similarly, if the plane cutting the mirror along line of longitude EO and containing R, U, O, and M is taken, it will intersect those parallel

planes along the two lines MO and RU. Therefore, these two lines are parallel. Hence, angle SOM = angle KUR, and angle MOI = angle RUY. But it has already been shown that angle KUR = [angle] RUY. Accordingly, angle SOM = angle MOI, so point S can be reflected to I from point O, barring interference from the cone.

[2.268] But it has already been shown [earlier in this proposition] that [the form of] point M can be reflected to I from point G, and so [the form of] point S, which lies on [straight] line SMG, can be reflected to I from point G. Therefore, [the form of] point S is reflected to I from two points on circle PG, which is impossible. It follows, then, that the main [supposition of this proposition], i.e., that [the form of] point B can be reflected to A from some point other than G on the mirror, is impossible, which is what was proposed.

[2.269] **[PROPOSITION 31]** Now, given a [convex] conical mirror, the point of reflection can be found [as follows].

[2.270] For instance, Let G [figure 5.2.31, p. 246] be the vertex of the cone, and at that point produce plane MNG parallel to the base of the cone. Let A be the visible point and B the center of sight. A and B will [both] lie above that plane [i.e., MNG]; or [they will both lie] below it; or [they will both lie] in the plane itself; or one [will lie] above it [while] the other [lies] below it; or one [will lie] in the plane [while] the other [lies] above or below it.

[CASE 1]

[2.271] Let them lie below the plane, and from point A produce a plane that cuts the cone parallel to the base, and from point G to point B draw a line, which, when extended, will fall on the plane produced from A [through the cone], since it lies between parallel planes [i.e., MGN and HEA]. Let H be the point where this line falls [on that plane].

[2.272] Now, it is proven according to the previous method [applied in proposition 30 above] that [the form of] A is reflected to H from some point on the circle formed on the cone by the plane [HAT] produced from points A and H. Let the point of reflection be found on that circle [by proposition 25], and let it be E. Then draw line AB, and draw the cone's line of longitude GE, as well as axis GT of the cone.

[2.273] From point E draw line ET to the center of the circle, and it will fall to the axis, and it will be orthogonal to the plane tangent to that circle at point E. Then, with lines AE and HE drawn in, it will bisect the angle formed by them, and it will divide line AH [in half]. Let R be the point of division.

[2.274] It is evident that GE and ET form a plane that cuts line AB. Let F be the point of intersection, and from point F draw normal FC to line GE,

and it will be perpendicular to the plane tangent to the cone along line GE. Then, from point A draw AL parallel to line FC. FC, moreover, will intersect the axis at point K. From point A draw AS parallel to line RT, and from point E draw common section EO of plane AEH and the plane tangent to the cone along GE. It will fall orthogonally to AS, since it is orthogonal to ER [to which AS is parallel, by construction].

[2.275] Draw line BC, which, when extended, necessarily intersects line AL [which is parallel to FC, by construction]. Let the intersection be at point L, and from point C draw common section CP of the plane tangent [to the cone along GE] and plane ABL. Draw lines LS and PO.

[2.276] It is obvious that plane ALS is parallel to plane GEK [because AS is parallel to RT, by construction, AL is parallel to FC, by construction, and FC and RT lie in the same plane], and lines CE and PO lie in the plane tangent [to the cone along GE], that plane intersecting those parallel planes [ALS and GEK] along the two lines CE and PO. Hence, CE is parallel to PO.

[2.277] Furthermore, draw line HE until it intersects AS at point S. It is clear that line ES lies in plane HEG, and BL lies in the same [plane], and this plane cuts the aforementioned parallel planes [ALS and GEK] along the two lines EC and LS. Therefore EC is parallel to LS. Therefore PO will be parallel to LS [since it is parallel to CE], so $AO:OS = AP:PL$ [by Euclid, VI.2].

[2.278] But it is clear that angle [of reflection] $HER = \text{angle [of incidence] } REA$ [by construction]. Angle $ESA = \text{angle } EAS$ [because angle of reflection $REA = \text{alternate angle } EAS$, and angle of incidence $HER = \text{alternate angle } ESA$], and EO is perpendicular [to AS, so] $AO = OS$ [by Euclid, I.26]. Accordingly $AP = PL$. And CP is perpendicular to AL, since it is perpendicular to FCK. Thus [triangle CAL is isosceles, so] $CL = CA$, and angle $CLA = \text{angle } LAC$. Accordingly, angle $BCF = \text{angle } ACF$ [because, given that FCK and APL are parallel, angle $BCF = \text{alternate angle } CLA$, and angle $ACF = \text{alternate angle } LAC$]. Therefore, A is reflected to B from point C, which is what was proposed.

[CASE 2]

[2.279] But if the center of sight and the visible point [both] lie in plane MGN [figure 5.2.31a, p. 247], let the former be at point M, the latter at point N, draw lines MG, NG, and MN, and bisect [angle] MGN with line UG. It is clear that [the form of] N is reflected from G to M.⁹⁰ It is also clear that line UG and the cone's axis lie in a plane that cuts the cone along a line of longitude [GE].

[2.280] From point U draw UE perpendicular to this line of longitude. Through point E produce the plane parallel to the base [of the cone], and this plane will cut the cone along a circle. Let ET be the common section of

plane UEG and this circle. It is evident that it will fall on the axis as well as on the center of the circle.

[2.281] Then, from point M draw a line [MH] parallel to GE, a line that falls at point H on the plane of that circle. Likewise, from point N draw a line [NA] parallel to GE and falling at point A [on the plane of the circle]. Draw AH, and let ET intersect it at point R.

[2.282] It is obvious that MH, which is parallel to GE, lies in the same plane with it, and this plane [MHEG] intersects plane MGN and plane HEA along the two lines MG and HE. Therefore [because planes MGN and HEA are parallel, by construction], MG is parallel to HE. By the same token, AN and GE lie in the plane [ANGE] that cuts those parallel planes [MGN and HEA] along NG and AE. Hence, NG is parallel to AE. Likewise, plane UGE intersects the same planes [MGN and HEA] along the two lines RE and UG. Thus, UG and MG are parallel to HE and RE [in reverse order], so angle MGU = angle HER, angle UGN = angle REA, and angle HER = angle REA. And so [the form of] point A can be reflected to H from point E.

[2.283] Accordingly, if a line is drawn from point A parallel to UE, if another [is drawn] parallel to RE, if ME is extended until it intersects the line parallel to UE, if the common sections are drawn, as before, and if the preceding proof is repeated, it will be clear that [the form of point] N can be reflected to M from point E.⁹¹ E will therefore be the point of reflection, which is what was proposed.

[CASE 3]

[2.284] Now, if both [the center of sight and the visible point] lie above MGN [figure 5.2.31b, p. 247], construct the cone opposite the original one. To do so, extend the lines of longitude of the previously constructed cone, and through point A pass a plane that cuts this latter cone parallel to the base, and it will cut the cone along circle YZ.

[2.285] B will either lie in this plane, or it will not. If it does, then carry out the procedure from point B. If not, then extend line GB until it intersects this plane. Let the intersection be at point D. It is evident that A is reflected to D from some point inside circle YZ.⁹² Find that point (as we will later prove and explain [in proposition 38 below], it is not among those on the anterior [surface of the cone]), and let it be Z. Lines DZ and AZ will be drawn, and let line PZ bisect the angle [formed by them].

[2.286] Extend line ZG to the other cone, and it will reach its surface to form a line of longitude [on it]. Let it be line ZGE. It is obvious that plane PZE will intersect line AB. Let it intersect at point Q, and from point Q draw a perpendicular to line GE, and let it fall at point E.⁹³ It will also be perpendicular to the plane tangent to the cone along line GE. Through

point E produce plane A'EH parallel to the base [of the cone], and from point D draw line DH parallel to ZE and intersecting that plane at point H. Let A'A be parallel to that same line.

[2.287] It is clear that DH is parallel to ZE, and they lie in the same plane [DHEZ], which intersects the parallel planes [A'EH and AZD] along the two lines DZ and HE. Accordingly, HE and DZ are parallel. Likewise, AZ and A'E are parallel. And it is evident that PZ passes through the center of circle YZ, and so does RET [pass] through the center of the other circle along which plane A'EH cuts the cone. Therefore, plane PZER intersects the two parallel planes [AZD and A'EH] along the two lines PZ and RE. PZ, then, is parallel to RE, so angle AZP = angle A'ER. And therefore angle A'ER = angle REH, so [the form of] A' is reflected to H from point E.

[2.288] Accordingly, if a line is drawn from point A' parallel to QE, if another [line is drawn] parallel to RE, and if the common sections [are drawn] as before, and if we repeat the previous method of proof, it will be evident that [the form of] point A is reflected to B from point E, which is what was proposed.⁹⁴

[CASE 4]

[2.289] If, however, the center of sight lies in the plane parallel to the [cone's] base and passing through its vertex, i.e., [point] G, and if the visible point lies below this plane, the point of reflection will be found in the following way.

[2.290] Let M [figure 5.2.31c, p. 248] be the center of sight, A the visible point, and MGN the plane parallel to the base of the cone. From point A produce a plane parallel to the base of the cone, and it will cut the cone along circle DEK with centerpoint T. From point M draw MH perpendicular to this plane, and draw line HT. From point A draw line AEQ to HT within the circle such that EQ = QT, according to the earlier account [in proposition 23, lemma 5]. Then, draw line TEI, and from point H draw HB parallel and equal to TE. Draw lines MB and BE. It is evident that plane GTE will intersect line AM. Let F be the point of intersection, and from point F draw FOC perpendicular to line GE and intersecting it at point O. Then, draw lines MO and AO. I say that O is the point of reflection.

[2.291] It is clear that HB is parallel and equal to TE [by construction]. Therefore, HT is parallel and equal to BE [since they are cut by equal and parallel lines HB and TE so as to form a parallelogram]. But MH is parallel and equal to GT, since both are perpendicular [to parallel planes DEK and MGN]. Accordingly, HT is parallel and equal to MG. Therefore, MG is parallel and equal to BE [since BE is parallel and equal to HT, by previous conclusions], so MB is parallel and equal to GE.

[2.292] It is also clear that angle QTE = angle QET [because QT = EQ, by construction, so triangle EQT is isosceles], and so it is equal to angle AEI [which is the vertical angle of QET]. But it is [also] equal to angle IEB [which is alternate to QTE]. Thus, [angle] IEB = angle IEA, so [the form of] A is reflected to B from point E. And since MB is parallel to GE, if a line parallel to FOC, as well as to TE, is drawn from point A, and if the earlier figure and proof [based on figure 5.2.31, p. 282] are repeated, it is clear that A is reflected to M from point O, and so [we have done] what was proposed.

[CASE 5]

[2.293] Now, if M lies in the plane [of MGN], and A lies above that plane, then the cone opposite the original one will be produced. Pass a plane through A that [cuts the opposite cone] parallel to its base, and the point of reflection is found among points lying inside the circle formed by this plane. Draw the line to G from that point, and extend it. And the point of reflection will be found according to subsequent analysis [in proposition 38 below], and the same method of proof will apply.⁹⁵

[CASE 6]

[2.294] But if the [relevant] points, i.e., the center of sight and the visible point, are so disposed that one of them lies above the plane of the [cone's] vertex and the other below it, let one of them be L [figure 5.2.31e, p. 249] and the other A, and let the plane at the vertex be MGN.

[2.295] Through point A extend a plane parallel to the base of the cone and intersecting it along circle DE with center T, and draw line LG. It will intersect plane AED. Let K be the [point of] intersection, and in circle DE find point E such that tangent SE drawn from that point bisects the angle formed by lines KE and AE [by proposition 22, lemma 4].

[2.296] Then, from point L draw a line [LB] parallel to [line of longitude] GE, a line that will necessarily intersect line KE. Let B be the intersection. It is clear that L lies in plane GEK, and LB, being parallel to GE, lies in that same plane. Draw line TEI. It is evident that plane GTE intersects line LA. Let it intersect at point U, and from there draw normal UOC to the plane tangent [to the cone at point O]. Finally, draw lines AO and LO.

[2.297] It is clear that [angle] AES = angle SEK [by construction], and since angle IES is a right angle, and [angle] SET is a right angle, [angle] IEA [which = IES + AES] = angle TEK [which = SET + SEK]. And therefore angle AEI = angle IEB [from which it follows that angle of incidence AET = angle of reflection TEB], so [the form of] A is reflected to B from point E. If, then, a line parallel to UO and a line parallel to IT are drawn from point A, and if the proof [from the previous cases] is repeated, it will be clear that [the form

of] A is reflected from point O to L, and so [we have done] what was proposed.⁹⁶

[2.298] It is therefore clear how the point of reflection can be found, and what has been discussed must be understood to apply to a single center of sight. But in [the case of] both eyes, the same thing happens, because the same form and the same location for the form is perceived by each eye, and, as has been claimed in the case of the convex spherical mirror, the forms perceived by both eyes in these sorts of mirror [i.e., conical convex] appear identical because of their proximity; sometimes they share precisely the same [image-]location, sometimes their [image-]locations overlap, and sometimes they are separated, but only a little bit.

[2.299] Furthermore, [any] form that reaches these mirrors along the normal returns along the same line, just as was shown earlier, and the form that is perceived by one eye along the perpendicular is perceived by the other eye along a line of reflection [oblique to that perpendicular]. Nevertheless, the locations of those forms are continuous [with one another], so the form appears identical to both eyes.

[Concave Spherical Mirrors]

[2.300] **[PROPOSITION 32]** In the case of spherical concave mirrors, the normal dropped from the visible point [to the mirror] sometimes intersects the line of reflection, and sometimes it is parallel to it. When it intersects, the image-location will sometimes lie on the [surface of the] mirror, sometimes behind the mirror, and sometimes in front of it. And when the image-location lies in front of the mirror, it will sometimes lie between the center of sight and the mirror, sometimes at the center of sight [itself], and sometimes beyond the center of sight. And we will demonstrate this [as follows].

[2.301] Let A [figure 5.2.32, p. 250] be the center of sight and D the center of the mirror, and through these points produce a plane that will cut the mirror along [great] circle HBFG. This plane will be a plane of reflection, because it is orthogonal to any plane that is tangent to the [great] circle [on the sphere]. Draw line AD, and from point A to the circle draw line AE, which is longer than AD. From point D to the circle draw [line] DH parallel to line AE, extend AD to points B and I [on the circle], and draw line DE.

[2.302] It is evident that angle AED is less than a right angle, because ED is a radius, and [by Euclid, III.31] any line in a circle forms an acute angle with the radius [or any other segment of the diameter]. At point E form angle DET equal to angle AED. It is clear that ET will fall inside the circle

[since $\angle DET$ is acute] and will intersect line DH . Let T be the point of intersection. It is also clear that $\angle ADE > \angle DET$, so ET will intersect AB . Let it intersect at point Z .

[2.303] Then, from point A draw line AN to arc EH , draw line DN , and at point N form with line NM an angle $[DNM]$ equal to angle DNA , and this line $[NM]$ will necessarily fall inside the circle [because it cannot be tangent at N] and will intersect DH . Let it intersect at point M . It is clear, as well, that AN will intersect DH outside the circle [since $\angle DAN$ and $\angle ADH$ sum up to less than two right angles]. Let L be the intersection.

[2.304] In addition, draw line AG from point A to arc EF , draw DG , and let $\angle AGD = \angle DGQ$. It is obvious that QG will intersect DH . Let Q be the point of intersection. It is also obvious that AG will intersect DH on the side of F . Let O be the intersection. Moreover, since the arc $[GY]$ that subtends GO within the circle is greater than arc GH , it is evident that GQ falls between D and H . For if line GH is drawn, angle $HGD[P]$ will be subtended by a greater arc [i.e., HP] than angle $[P]DGA[Y]$, i.e., arc PY .

[2.305] Furthermore, from point A draw line AC to arc FB so as to cut DH at point S in such a way that $CS > SD$, and draw DC . It is clear that angle DCA is acute. Form angle DCK equal to it. Since $\angle CDS > \angle DCS$, it is evident that CK will intersect DH . Let K be the point of intersection.

[2.306] According to the foregoing construction [making $\angle DET = \angle AED$], it is clear that [the form of] point T propagates to E and is reflected to A . TD is the normal dropped from point T , and this line, which is perpendicular to the plane tangent to the circle [at H], is [by construction] parallel to AE , the line of reflection, so it will not intersect it.

[2.307] [The form of] point Z , on the other hand, propagates to E and is reflected to A . AZ is the normal dropped from point Z , and it intersects AE at point A , so the image-location for point Z will be [at center of sight] A .

[2.308] On the other hand, [the form of] point M propagates to N and is reflected to A . Normal MD dropped from point M intersects [line of reflection] AN at point L , which lies behind the mirror, and the image-location for point M will be L .

[2.309] The form of point Q , however, propagates to G and is reflected to A , and its image-location [where line of reflection AG intersects normal QD] will be O , which lies beyond the center of sight.

[2.310] Finally, the form of point K propagates to C and is reflected to A , and the normal [dropped] from it is KD , and [so] its image-location is S [where KD intersects line of reflection CA].

[2.311] From the foregoing it is thus clear that some of the images lie behind the mirror, some between the center of sight and the mirror, some at

the center of sight itself, and some beyond the center of sight, which is what was proposed.

[2.312] It is evident, moreover, that the visual faculty grasps forms that face it, so, when the image-location lies behind the mirror or between the eye and the mirror, the image is grasped according to how it is actually located. However, when the normal dropped from the visible point is parallel to the line of reflection, the image will appear at the point of reflection. For, since that [visible] point is [a] sensible [spot] represented by the point imagined [to lie] at its center, the image of any portion of that sensible spot taken beyond the midpoint [toward the mirror] will lie behind the mirror, whereas the image of the portion in front of the midpoint [away from the mirror] will lie between the eye and the mirror, and since the whole form appears as a continuum composed of the parts behind and the parts in front [of the mirror], the form of that sensible spot will necessarily appear in an image-location on the mirror itself.⁹⁷

[2.313] But in the case of images that are located at the center of sight, they are not grasped according to how they are actually located, so [visual] deception often occurs in these sorts of mirrors [i.e., spherical concave]. To make this clear, stand a wooden rod less than half the radius of the mirror in length upright upon the surface of the mirror [along a normal]. Let the center of sight be situated directly above this rod, and look at a spot on the mirror that lies farther from the rod than the center of sight does [from the mirror] along the normal passing through the rod. The image of this rod will appear behind the object itself, but it will not be perceived properly; on the contrary, it will appear bowed when it is not.⁹⁸ Hence, in these sorts of mirrors the image is seen according to how it is actually located only when the image-location lies behind the mirror or between the eye and the mirror. But when the center of sight lies on the normal passing through the rod, it does not perceive the form of that rod clearly.

[2.314] On the other hand, if the center of sight is located on a diameter of the sphere and at its center, then, since every line dropped from it to the mirror is perpendicular to the mirror, the form of no point will be perceived except a point within the portion of the circle lying between the edges of the visual cone that is imagined to extend [to the mirror's surface] from the center of the circle [where the center of sight is located]. For the form of any other point will fall on the mirror along an oblique line, and it is necessarily reflected along an oblique line, so the line of reflection will not pass through the center, and so it will not reach the center of sight.⁹⁹

[2.315] However, if the center of sight lies on a diameter but not at the center, it will not perceive the form of any point on the radius within which it lies. For the angle that the two lines from the given point on the radius to

the center of sight will form at the same point on the mirror will not be bisected by the normal extended from that point on the mirror, since that normal is directed to the center of the mirror. But it can perceive the form of any point on [a] radius other [than the one opposite it on the diameter upon which it lies].¹⁰⁰

[2.316] **[PROPOSITION 33]** Now, when a point is viewed in this sort of mirror, if the normal is not parallel to the line of reflection, the line extended from the center of the mirror to the visible point will have to the line extended from that same centerpoint to the image-location the same ratio that the line drawn from the visible point to the point we have called the [end]point of tangency has to the line extended from the [end]point of tangency to the image-location.

[2.317] For example, let E [figure 5.2.33, p. 251] be the center of the mirror, B the visible point, A the center of sight, G the point of reflection, and ZG the line of tangency. Either ZG will intersect EB, or it will be parallel to it.

[2.318] Let it intersect at point T. Line EB will intersect AG, but not at point G, since EB and BG are two [distinct] lines. Therefore, they will intersect behind G, or between G and A, or at A, or in front of A. Let [the intersection be] behind G, at point H. Accordingly, I say that $EB:EH = BT:TH$.¹⁰¹

[2.319] Draw normal EG, and from point H draw [line HL] parallel to line BG, so it will intersect EG. Let L be the intersection, and from point B draw [line BQ] parallel to GH, so it will necessarily intersect ZT. Let Q be the intersection.

[2.320] It is evident that angle BGE = angle AGE [by construction]. But angle BGE = [alternate] angle GLH, and angle AGE = [vertical] angle LGH [so triangle LGH is isosceles]. Therefore, LH = GH. Likewise, angle BGQ = [angle] AGZ [by construction], and angle AGZ = [alternate] angle GQB [making triangle GBQ isosceles], and so BQ = BG, from which it follows that $BG:HL = BQ:HG$ [because BG = BQ, and HL = HG].

[2.321] But since angle GHT = [alternate] angle TBQ, triangle TBQ will be similar to triangle GHT [because they have two equal corresponding angles, i.e., angles TBQ and GHT and angle BTQ and vertical angle GTH]. Thus, $QB:HG = BT:TH$ [by Euclid, VI.4], and so $BG:HL = BT:TH$ [because we have established that $BG:HL = QB:HG$, and $QB:HG = BT:TH$]. But since triangle BGE is similar to triangle HEL [HL and GB being parallel, by construction], then $BG:HL = EB:EH$, and so $EB:EH = BT:TH$, which is what was proposed.

[2.322] The same proof will apply if the image-location [H] lies between A and G [figure 5.2.33a, p. 251], at A [figure 5.2.33b], or beyond A [figure 5.2.33c, p. 252].

[2.323] If, however, line of tangency ZG is parallel to normal BH [figure 5.2.33d, p. 252], draw perpendicular GE, which, since it is perpendicular to ZG, will be perpendicular to BH. And [right] angle BEG = [right] angle HEG, and angle BGE = angle EGH. It follows that triangle BGE is similar to triangle EGH. Therefore, BE:EH = BG:GH, which is what was proposed, because in this case, no [end]point of tangency other than G can be assumed, according to the way we defined the [end]point of tangency earlier [in proposition 6, where the endpoint of tangency lies at the intersection of the normal dropped from the object-point and the line tangent to the point of reflection].

[2.324] **[PROPOSITION 34]** Now, let DGT [figure 5.2.34, p. 253] be the [great] circle [cut from the mirror by the plane of reflection], A the center of sight inside the circle, E the center of the mirror, and B the visible point. Draw diameter DAG.

[2.325] If B lies on radius EG, there can be reflection from some point on semicircle GTD as well as from some point on the semicircle opposite [and below] it. For if a line is drawn to some point [F] on semicircle GTD from some selected point [B] on radius EG, and if another line is drawn from point A to the same point [F on semicircle GTD], those two lines will form an angle that the radius [EF] extended from point E will bisect at that point. The same holds in the case of the opposite semicircle.

[2.326] But if B lies outside diameter DAG [figure 5.2.34a, p. 253], draw the diameter passing through B, and let it be TBQ. I say that [the form of] B can be reflected to A from the arc lying between the radii in which A and B lie, and likewise from the arc opposite it, i.e., from arc TD and from arc GQ, but it cannot be reflected from any point on arc GT or arc QD.

[2.327] For instance, take some [potential] point [of reflection] K on arc GT, and draw lines AK and KB until KB intersects diameter DG at point O. Since O and A lie on the same side of E, which is the center of the circle, the normal dropped from point K to E will not divide angle OKA, and so [the form of] B is not reflected to A from point K. Likewise, if another [potential] point [of reflection] F is chosen [on that same arc], it will be obvious that normal EF will not divide angle AFB, and so [the form of] B is not reflected to A from point F.

[2.328] That reflection can, however, take place from a point on arc TD or on arc GQ is clear from the following. Let M be a [potential] point [of reflection] on arc DT, and draw lines AM and MB to form quadrilateral AMBE. Therefore, normal EM will divide angle AMB.

[2.329] By the same token, let H be a [potential] point [of reflection] on arc GQ. Line AH will intersect diameter TQ at point C, and line HB [will

intersect] the same [diameter] at point B. And these two points lie on different sides of the centerpoint [E], so line EH will divide that angle.

[2.330] Likewise, if B lies on the surface of the mirror, or outside the mirror, as long as A lies inside the mirror, the same method of proof will apply as before. And the same holds if A lies on the surface of the mirror, and B lies inside or outside it.¹⁰²

[2.331] But if AP is drawn from A parallel to TE [figure 5.2.34b, p. 254], the locations of images reflected from points [such as M] on arc TP will lie outside [and behind] the mirror, whereas the locations of images [reflected from points such as N] within arc PD will lie [at points, such as S] beyond the center of sight A, and the locations of images [reflected from points such as H] within arc QG lie [at points, such as X] between the center of sight and the mirror. And the same thing that has been just said about image-locations must be understood [to apply] when AM is drawn parallel to line TQ.

[2.332] On the other hand, if A lies outside the mirror and B inside it [figure 5.2.34c, p. 254], what we claimed will be evident [as follows]. From point A draw lines AH and AZ tangent to circle GTD, draw the two diameters AEG and TEQ, and let B lie on diameter TEQ. [The form of] B is reflected to A from some point on arc TD, but it is obvious that [it is] not [reflected] from any point on arc ZD. Therefore, [it is reflected] from some point on arc TZ, and likewise from some point on arc GQ opposite TD. However, according to the previous method [of demonstration], reflection will not occur from arc TG or [arc] DQ.

[2.333] If, however, B lies outside this diameter [TEQ] and on another, which is likewise [designated] T'E'Q', reflection will occur from arc T'D, but only within portion T'Z on it, and [it will occur] from G'Q', the arc opposite it. Reflection will not occur, however, from arc T'G or [arc] D'Q'.

[2.334] **[PROPOSITION 35]** Furthermore, if a diameter is taken on a [great] circle within a spherical concave mirror, any point on that diameter, no matter how far it is extended, can be an image-location.

[2.335] For instance, let AG [figure 5.2.35, p. 255] be a diameter of circle AMG, whose center is D. Take point Z on this diameter, [and take] E [as] the center of sight. I say that Z can be an image-location.

[2.336] For instance, draw line ETZ, T being a point on the circle. Draw line DT. Angle ETD will be acute. Form [angle] DTL equal to it. It is clear [from the equality of angles ETD and DTL] that [the form of] L is reflected to E from point T and that its image will be Z.

[2.337] Similarly, if point L is selected, it will be clear that it is an image-location. Draw line EL to point B on the circle, and draw line DB. Angle EBD will be acute. Form [angle] DBC equal to it. [Accordingly, the form of]

point C is reflected to E from point B, and its image-location will be L, and so whatever other point is chosen, the same proof will hold.

[2.338] Now, among the points that are perceived in these sorts of mirrors, some of their images are distributed in four locations, some in three, some in two, and some in [only] one. A point whose image lies at four locations is reflected from four distinct points, and from no others nor from more [than four]. A point whose image occupies three locations is reflected from three points on the mirror, and from no more than three; one whose [image occupies] two [locations is reflected] from two points [only]; whereas one whose image lies in a single location may have its reflection occur from one point only, and it may [have it occur] from some point on a specific [great] circle, but from no other [point within the mirror].

[2.339] **[PROPOSITION 36]** For example, let E [figure 5.2.36, p. 255] be the center of sight, let H be a visible point on the same diameter, and let D be the center of the circle. Draw diameter ZEHA. ED is either equal to DH, or not.

[CASE 1]

[2.340] Let it be equal, and from point D on EH draw diameter GDB perpendicular [to EH], and draw lines HG, GE, HB, and BE. It is evident that triangle HGD = triangle EGD, and it equals triangle HBD, as well as triangle EBD. Since angle HGE is bisected [by diameter GDB], it is clear that [the form of point] H is reflected from point G to E, and its image-location is E [where line of reflection GE intersects normal HD dropped from visible point H]. Likewise, [the form of point] H is reflected to E from point B, and its image-location is E.

[2.341] Therefore, if diameter ZEHA remains stationary and semicircle AGZ, or only triangle HGE, is rotated [around AZ, as an axis] throughout [the circumference of] the sphere, point G will describe a circle in the course of its motion, and from any point on that circle [the form of point] H is reflected to [point] E, and its image-location will always be point E, and so [we have demonstrated] what was proposed.

[2.342] That reflection of [the form of] point H to E cannot take place from any point other than one on that circle [of revolution] is evident from the following. Take point C. Line EC > line EG, and line HC < line HG, so EC:HC $\neq$ ED:DH [which means that angle ECD $\neq$ angle DCH, by Euclid, VI.3]. Therefore, line DC will not bisect angle ECH, so [the form of point] H cannot be reflected to E from point C. The same disproof will apply if C is taken between G and Z.

[CASE 2]

[2.343] But if $ED > DH$, adjust the figure [accordingly], and add line HQ to line EH [figure 5.2.36a, p. 256] so that $EQ, QH = DQ^2$. Accordingly $EQ:DQ = DQ:HQ$ [by Euclid, VI.17], so $EQ:DQ = ED:DH$, as Euclid demonstrates [in V.19].¹⁰³

[2.344] Produce a circle with a radius of QD having its center at Q, let G and B be the points where the two circles intersect, and draw lines EG, EB, QG, QB, DG, DB, HG, and HB. Accordingly, it is evident that $EQ:QG = QG:QH$ [because radius QD = radius QG, and, as established earlier, $EQ:QD = QD:QH$], and angle GQH is common to both triangles EQG and HQG [whereas angles QHG and QGE are right]. Therefore, those two triangles are similar. So [by Euclid, VI.4] $EQ:QG = EG:GH$. Therefore, $ED:DH = EG:GH$ [since we established that $EQ:QG = EQ:QD = ED:DH$], so line DG will bisect angle EGH [by Euclid, VI.3].

[2.345] Consequently, [the form of] point H is reflected to E from point G, and its image-location is point E. Likewise, [the form of point] H is reflected from point B to E, and its image-location is E.

[2.346] Hence, if points E and H are held stationary and triangle EHG is rotated [around QE, as axis], point G will describe within the sphere a circle from any point of which [the form of] H is reflected to E, and E will always be the image-location.

[2.347] Moreover, as before, it is clear that [the form of point] H cannot be reflected to point E from any point [within the sphere] other than one on that circle. For if C is taken between G and A, $EC > EG$, and $HC < HG$, so $EC:HC \neq ED:DH$, and so [by Euclid, VI.3] DC does not bisect angle ECH. Likewise, if C is taken between G and Z, [then the supposition that reflection can occur from C] can be disproved.

[2.348] And so what was set out [has been demonstrated], as long as it is borne in mind that E is an imaginary point, and the circle [of revolution] with E as pole is an imaginary circle, and H is an imaginary point. Therefore, what has been claimed according to geometrical demonstration is not to be understood according to ocular proof, because imaginary objects are invisible to sight. But since the form of H appears continuous with the forms of other points [in close proximity within the visible spot represented by H], the form that sight will see is a form whose midpoint is H, and the midpoint of that form [when reflected to E] will be E, and this form will be reflected from a circular segment of the mirror within which the aforementioned circle [created by the rotation of G] will represent the midline with E as its pole.

[2.349] Moreover, if $ED > DH$, insofar as it can be enough longer that [the form of point] H does not reflect to E from point G, it must be under-

stood that, unless $EA:AH > ED:DH$, [the form of] H cannot be reflected to E.

[2.350] For if it could be reflected, let it be reflected from point G. Angle GDH will be less than a right angle, since it is subtended by an arc less than a quarter of the circle. From point G draw the tangent, which will necessarily intersect EA. Let Q' be the intersection. By the [33d proposition]¹⁰⁴ we know that] $EQ':Q'H = ED:DH$, but $EA:AH > EQ':Q'H$. Therefore, $EA:AH > ED:DH$, and so, by necessity, if [the form of point] H is reflected to E, $EA:AH > ED:DH$.¹⁰⁵ The claims that have been made are therefore obvious when the center of sight and the visible point lie on the same diameter.

[2.351] **[PROPOSITION 37]** Furthermore, if the visible point and the center of sight do not lie on the same diameter, and if they lie outside the mirror, the [form of the] visible point is reflected to the center of sight from only one point on the mirror.

[CASE 1]

[2.352] For instance, let T [figure 5.2.37, p. 257] be the visible point, H the center of sight, and D the center of the sphere, and draw lines HD and TD. Plane HDT intersects the sphere along circle EBQG.

[2.353] It is clear that [the form of] T is not reflected to H except from some point on this circle. According to earlier discussion [in proposition 34 above], it is also clear that it is not reflected from arc QG or BA. It is therefore reflected from either arc GB or AQ.

[2.354] Bisect angle TDH with line LEDZ, and draw tangent KEF from point E. If points T and H lie on that tangent, [the form of point] T will not be reflected to H from any point on arc BG. For if a line will be drawn from point T to some point on this arc [on the] inner [side of the circle], the line drawn from point H to the same point will fall to it on the outer side [of the circle], not the inner side, and so there will be no reflection.¹⁰⁶

[2.355] Moreover, it will be clear from the following that reflection may occur from only one point on arc AQ. Draw lines TZ and HZ. Since angle TDH is bisected, [angle] $TDZ = \text{angle } HDZ$.

[2.356] Either lines TD and HD are equal, or they are not equal. If they are equal, and if line DZ is common, then triangle TDZ = triangle HZD, and angle TZH is bisected by line DZ, and so [the form of point] T is reflected to H from point Z.

[2.357] That it cannot [be reflected] from another point [on arc AQ] will be established as follows. Take point O, draw lines TO and HO, and let line ODM bisect the angle [formed by them]. It is clear [by Euclid, III.8] that $TZ < TO$ and that $HO < HZ$, and [by Euclid, VI.3 it is also clear that] $TZ:HZ = TL:LH$ [since angle HTZ is bisected by ZL, leaving triangles HZL and TZL

equal] and $TO:HO = TM:MH$ [since angle TOH is supposedly bisected by OM , leaving triangles HOM and TOM equal]. But $HO:TO < HZ:TZ$ [since $TO > HO$, and $HZ = TZ$]. Accordingly, $HM:MT < HL:LT$, which is impossible [because, according to the bisection of angle HOT by HM , $HM:MT = HO:TO$, whereas, according to the bisection of angle HZT by ZL , $HZ:TZ = HL:HT$, so it should necessarily follow that $HM:MT = HL:LT$, because M and L should cut equiproportional segments from HT].

[2.358] It is therefore obvious that, if T and H lie the same distance from the center [of the mirror], and if they lie above the tangent, [the form of point] T is reflected to H from only one point on the mirror, and it will have a single image-location.¹⁰⁷

[CASE 2]

[2.359] Now, let BDQ and ADG [figure 5.2.37b, p. 258] be two diameters within the sphere [from which the mirror is formed], let diameter EDZ bisect angle BDG , and from point E draw the two [lines] ET and EH perpendicular to the two diameters BD and GD .

[2.360] It is obvious that triangle $ETD =$ triangle EHD , since ED is common to both [and angles EHD and ETD , as well as EDH and ETD are equal, by construction]. Thus, [the form of point] T is reflected to H from point E . By the same token, [it is reflected to point T] from point Z . It is also obvious [from proposition 34 above] that it is not reflected to [point] E from any point on arc AB or arc GQ , nor is it reflected from any point on arc AQ other than from point Z , according to the previous demonstration [in case 1 above]. But that it cannot be reflected from any point on arc BG other than from point E will be shown as follows.

[2.361] Given point O [from which reflection supposedly does occur], draw lines TO , HO , and DO . Construct a circle the size of DE [as diameter] that passes through the three points T , D , and H , and line DE will be the diameter [by Euclid, III.31], since the angle ETD that it subtends is a right angle [by construction]. Therefore, that circle will pass through point E .

[2.362] Accordingly, since E is common to each circle [i.e., $TDHE$ and $AZGE$], and since it lies on the same diameter [within each circle], the smaller circle will be tangent to the larger circle at point E , as Euclid demonstrates [in III.11]. Therefore, this [smaller] circle will intersect line DO .

[2.363] Let it intersect at point I , and draw lines TI and HI . It is already evident that $TD = DH$ [since they are corresponding sides of equal triangles TED and HED]. Thus, angle $TID =$ angle DIH , since they are subtended by equal arcs [DT and DH in circle HTE]. It follows that [being adjacent to angle TID] angle $TIO =$ angle HIO [which is adjacent to angle DIH]; angle $IOT =$ angle IOH , by construction, and IO is common. [Hence] triangle TIO

= triangle HIO [by Euclid, I.26], and $TO = HO$, which is impossible, since [by Euclid, III.7] $HO > HE$, [while] $TO < TE$, and $TE = HE$, as has been proven earlier. It therefore follows that [the form of point] T may not be reflected to H from any point other than E or Z.

[2.364] Moreover, from point E draw line EM to diameter TD, from line HD cut off a segment ND equal to MD, and draw EN and EM. It is evident that angle EMD is greater than a right angle [because it is exterior to right angle ETM in triangle ETM]. Cut from it [an angle] equal to a right angle with line CM, which will intersect DE. Let C be the point of intersection, draw NC, and construct a circle according to the size [of diameter] CD that passes through the three points M, D, and N. Since CMD is a right angle [by construction], CD will be the diameter, and the circle will pass through C. It is therefore obvious that [the form of point] M is reflected to N from point E [because triangle NCE = triangle MCE, so angle NED = corresponding angle MED], and likewise [it is reflected] from point Z, but from no other point on arc AB or QG; and it is obvious that [it does not reflect] from any point on arc AQ other than point Z.

[2.365] That [it does not reflect] from any point on arc BG other than from point E is evident from the earlier procedure. For if [some such] point is taken, if lines are drawn to that point from points T, D, and H, if a point is taken where the last circle will cut the diameter, and if lines are drawn to points T and H from the point of intersection, the same disproof as before will apply.

[2.366] From the foregoing, then, it is clear that, if a third diameter bisects the angle formed by two [other] diameters, and if from the endpoint of that [third] diameter perpendiculars are drawn to those [other two] diameters, the points on the diameters where those perpendiculars fall are reflected to one another from only two points on the mirror. In addition, any of the points taken on the perpendiculars to the diameters below these endpoints, i.e., toward the center, is reflected from only two points [on the circle], and the one is reflected to the other that lies equidistant from the center, and for all such points the image-location is twofold.¹⁰⁸

[CASE 3]

[2.367] Furthermore, given the two diameters BQ and AG [figure 5.2.37c, p. 259], and given that EZ bisects the angle formed by them, take point T on BD beyond the point [between D and T] where the perpendicular dropped from point E falls [upon DB], take DH on DG equal to DT, and draw TE and HE. [Thus, according to the preceding case, the form of point] T is reflected to H from point E, and likewise from point Z, [but] not from any other point on arc AQ, or from any point on arcs AB or GQ.

[2.368] Then, from point T on TD draw a perpendicular, which will intersect DE outside the circle on the [mirror's defining] sphere, because angle DTE is acute. Accordingly, let it intersect at point O, and draw lines TO and HO. Construct a circle passing through the three points T, D, and H. That circle will necessarily pass through point O, and DO will be its diameter. Draw lines TO and HO, and draw line KE tangent to circle BZG at point E. It is clear that the latter circle will intersect the first one, i.e., BZG, at two points. Let those points be L and M, and draw lines TL, HL, LD, TM, DM, and HM.

[2.369] Therefore, since arc TD = arc HD [by construction], angle TLD = angle DLH [by Euclid, III.27], and so [the form of point] T is reflected to H from point L. Likewise, angle TMD = angle DMH, and so [the form of point] T is reflected to H from point M. It is therefore evident that [the form of point] T is reflected to H from four points, i.e., E, Z, L, and M, and its image-location will be fourfold.

[2.370] But [the form of point] T cannot be reflected to H from any point other than these. For, let F be given [as such a point], draw lines TF, HF, and DF, extend DF until it intersects tangent KE, and let K be the intersection. Then draw lines TK and HK. Hence, angle TFD = angle DFH, by construction; and it follows that [being adjacent to angle TFD] angle TFK = angle KFH [which is adjacent to angle DFH]. But angle TKF[D] = angle [D]FKH, because they are subtended by equal arcs, and FK is common. [Therefore] triangle [TKF] = triangle [HFK], and so TK = KH, which is impossible, because HK > HO [since arc HK > arc HO, whereas] TK < TO, yet TO = HO [by construction].

[2.371] It is evident, then, that reflection does not occur from any point other than from the four [previously defined] points on the circle.

[2.372] [In summary] therefore, if two points, i.e., T and H, are taken on different diameters [at locations on those diameters that are] equidistant from the center, and if they lie at the points on the diameters where the perpendiculars drawn from the end of the diameter that bisects the angle formed by those two diameters fall, or if they lie between the center and those points, i.e., below the perpendiculars, as long as they lie equidistant from the center, then [the form of point] T will be reflected to H from two points only [as demonstrated in case 2 above].

[2.373] If, however, T and H lie at spots beyond the perpendiculars up to the interior surface of the circle, [the form of point] T will be reflected to [point] H from four points. But if they lie on the circle or outside it, but still below tangent KE, [the form of point] T will be reflected to H from two points only [i.e., E and Z]. And if they lie upon the tangent, [the form of point] T will be reflected to H from only one point [i.e., Z].¹⁰⁹ And these

phenomena occur as long as T lies the same distance as point H from the center.

[2.374] **[PROPOSITION 38]** Now, if T and H lie on different diameters, and if they lie unequal distances from the center, they will be reflected to one another from one point [only on the opposite arc].¹¹⁰

[2.375] For instance, draw diameters ADG and BDQ' [figure 5.2.38, p. 260], and let EZ bisect the angle formed by them. Let [object-point] T be nearer centerpoint D than [center of sight] H. Take [some arbitrary] line LQ and cut it at point M so that $QM:ML = HD:DT$ [by Euclid, VI.10]. Bisect LQ at point N, from point N draw perpendicular NK, and at point L form with line FL an angle [FLQ] equal to half of angle ADT. Angle FLQ will be acute, so FL will intersect NK. Let it intersect at point F, and from point M draw a line to side FL intersecting side NK at point K. Let that line cut side FL at point C such that $KC:CL = HD:DZ$ [by proposition 21, lemma 3 above].

[2.376] Then, on point D form angle IDA equal to angle LCM, and let I be a point on the circle lying above or below Z. At point I form angle O'ID equal to angle CLM, to the resulting line O'I drop perpendicular HC' from point H, draw line C'F' equal to line C'I, and draw lines HF' and HI.

[2.377] According to previous arguments, it is clear that from point M no line other than line MCK can be drawn to side FL to cut it in the way that line MCK does. For if [such a line] could [be found], let it be MPO. It is evident that $PO < CK$, which will become evident if line PY is drawn parallel to CK, that line being shorter than CK and longer than PO. But $PL > CL$. Thus, $PO:PL \neq KC:CL$, so $PO:PL \neq HD:DT$. It therefore follows that no line like MCK, other than MCK [itself], can be drawn from point M [to side FL].

[2.378] Now, since [angle] O'DI = angle LCM [by construction], and angle O'ID = angle CLM [by construction], triangle CLM will be similar to triangle IO'D. Therefore angle IO'D = [corresponding] angle LMC. It follows that [being adjacent to angle IO'D] angle C'O'H = angle KMN [which is adjacent to angle LMC], and right angle HC'O' = [right] angle KNM [both being right angles by construction]. [So] it follows that angle NKM [in triangle NKM] = angle C'HO' [in triangle C'HO', so triangles NKM and C'HO' are similar, by Euclid, VI.4].

[2.379] Now, if line DI is extended until it intersects HC' at point R, angle RDH = angle KCF [because angle KCF = vertical angle LCM, angle RDH = vertical angle O'DI, and angle LCM = angle O'DI, by construction]. Triangle RDH will [thus] be similar to triangle CKF [since corresponding angles RHD and CKF are also equal, by previous conclusions]. Therefore, $RD:DH = FC:KC$. But $HD:DI = KC:CL$ [by construction]. So $RD:DI = FC:CL$ [by Euclid, V.22]. Accordingly, $RI:DI = FL:CL$ [by Euclid, V.18]. But $DI:IO'$

= CL:LM, since triangle DIO' is similar to triangle CLM [by previous conclusions]. Therefore, RI:IO' = FL:LM [by Euclid, V.22]. But RI:IC' = FL:LN, because triangle RIC' is similar to triangle FLN [since right angle RC'I = right angle LNF, and angle RIC' = corresponding angle FLN]. Hence, IO':IC' = LM:LN [by Euclid, V.22]. So QM:LM = F'O':IO'.¹¹¹

[2.380] Now, if line UI is drawn from point I parallel to HF', and if line DA is extended until it intersects UI at point U, triangle O'UI will be similar to triangle HO'F'. Therefore, HO':O'U = QM:ML [because HO':O'U = F'O':O'I], and so HO':O'U = HD:DT [because QM:ML = HD:DT, by construction]. But since triangle HC'I = triangle HC'F', because HC' is perpendicular [to equal bases F'C' and C'I], then angle HF'C' = angle C'IH, so [angle] C'IH = angle UIO', and so [by Euclid, VI.3] HO':O'U = HI:UI [because angle UIH in triangle UIH is bisected by IO', insofar as angle UHI is composed of equal angles UIO' and C'IH], and so HI:UI = HD:DT.

[2.381] But angle UID > angle DIH [since angle UID > angle UIO', which = angle O'IH, of which DIH is a part]. From it [i.e., angle UID] cut angle PID equal [to DIH], and draw line TP. Let P be a point on diameter DA.

[2.382] It is evident that the ratio HI:UI is compounded from HI:IP and IP:UI, while HI:IP = HD:DP [by Euclid, VI.3], because DI bisects angle PIH [in triangle PIH]. Therefore, HI:UI, which is as HD:DP, is compounded of the ratios HD:DP [which = HI:IP] and PI:UI. But the ratio HD:DT is compounded of HD:DP and DP:DT. Thus, DP:DT = PI:UI.¹¹²

[2.383] Now, angle O'IH is half of angle UIH [by previous conclusions], whereas angle DIH is half of angle PIH [by construction]. It follows that angle DIO' is half of angle PIU, whereas angle DIO' is half of angle TDP, since it is equal to angle FLM. Therefore, angle PIU = angle TDP,¹¹³ and DP:DT = PI:UI [by previous conclusions]. Hence, triangle UIP is similar to triangle TPD [by Euclid, VI.4], and [so] angle UPI = [angle] TPD. TPI will thus be a straight line, because angle DPT + angle TPO' = two right angles, and so angle O'PI + angle O'PT = two right angles [since O'PI is alternate to DPT]. So [the form of point] T is reflected to [point] H from point I, and this proof will apply whether T lies outside or inside the circle, and [it will apply] likewise if point H is taken outside or inside [the circle], as long as [they are] unequally distant from the center.¹¹⁴

[2.384] **[PROPOSITION 39]** Now, having produced diameters BQ and AG, and [given] diameter EZ bisecting angle BDG, I say that, no matter what point other than Z is taken on arc AQ, an infinite number of point-pairs that are not equidistant from the center can be reflected [to one another] from that point.

[2.385] For example, take point H [as the potential point of reflection in figure 5.2.39, p. 261], and take point L on diameter GD. On diameter BD cut off [segment] MD equal to LD, and draw lines LM, LH, MH, and DH. Let F be the point where EZ cuts LM; [and so] $LF = FM$ [since triangles FMD and LDF are equal, by Euclid, I.4].

[2.386] Draw [line] HD until it falls on LM at point N. Hence, $LN < MN$. But, since angle MDF = [angle] FDL, as well as [vertical] angle QDZ, since angle MDA = [vertical] angle LDQ, and since angle ADH = [vertical] angle NDL, then angle LDH > angle MDH.¹¹⁵ Therefore [by Euclid, I.24], $LH > MH$, since MD and DH = LD and DH [respectively]. Thus, angle DHL < angle DHM, for if they were equal [then DHN would bisect angle MHD, and so, by Euclid, VI.3], $LH:MH$ would be as $LN:NH$, which is impossible. If, on the other hand, it were greater [i.e., if $DHL > DHM$], then cut from it [an angle] equal [to DHM], and it will be disproven in this [same] way.¹¹⁶ Therefore it is smaller [i.e., $DHL < DHM$].

[2.387] Accordingly from angle MHD cut off [angle] THD equal to it [i.e., DHL]. [The form of point] T is therefore reflected to [point] L from point H, and $TD < LD$ [i.e., T and L lie unequal distances from D].

[2.388] So, too, if points other than L and M that lie equidistant from point D are selected on diameters HD and GD, it is proven in a similar way that from point H there occurs a mutual reflection of points that lie at different distances from the center. And so the proof will be the same for an infinite number of points chosen on these diameters, as well as for any point chosen on arc AQ other than point Z [because, in order for reflection to occur from Z, the points must be equidistant from the center, as shown in proposition 37].¹¹⁷

[2.389] **[PROPOSITION 40]** Moreover, if T and L [figure 5.2.40, p. 261] are taken on [different] diameters at unequal distances from the center, and if they are reflected to one another from point H, [the form of point] T will not be reflected to [point] L from any point on arc AQ other than from point H.

[2.390] For if [it were reflected] from another [point], let it be K, and draw TK, LK, DK, LT, TH, LH, and NDH. Then extend DK until it falls to point C on LT. It is clear that $LH:TH = LN:NT$ [by Euclid, VI.3].

[2.391] Likewise, since angle TKC = angle LKC, by supposition, then $LK:TK = LC:CT$. But [by Euclid, III.7] $LH > LK$, and $TH < TK$. Hence $LH:TH > LK:TK$, so $LN:NT > LC:CT$, which is patently impossible. It follows that [the form of point] T cannot be reflected to L from any point on arc AQ other than from point H. What pertains to arc AQ is therefore evident.

[2.392] **[PROPOSITION 41]** To continue, let A [figure 5.2.41, p. 262] be the center of sight, B the center of the mirror, and draw diameter DABG. Take the plane in which AB lies so that it will somehow intersect the sphere along [great] circle DLG. I say that points that do not lie the same distance from the center as A are reflected [to A] from any point on semicircle DLG.

[2.393] For instance, take point E, and draw lines EA and EB. It is obvious that angle AEB will be acute, because it will be subtended by a smaller arc than a semicircle. Form [angle] OEB equal to it, and draw line OE as far as you like. It is evident that [the form of] every point on that line is reflected to A from point E.

[2.394] Now, if a perpendicular is dropped from point B to line OE, that perpendicular will be equal to, longer than, or shorter than BA. If it is equal, then every line, other than the perpendicular, that is drawn from point B to line OE will be longer than line BA, and so, with one exception [i.e., point O], every point on line OE will lie a different distance from the center than point A.

[2.395] If the perpendicular is longer [than AB], then all the points on that line [i.e., OE] will lie farther from the center than point A [and will therefore lie different distances from B than A]. But if the perpendicular is shorter, two lines equal to BA can be drawn from [points on] opposite sides of the perpendicular, whereas all the remaining lines will be either shorter or longer [than BA and will therefore lie at different distances from B]. It is therefore evident that [the forms of] points that do not lie the same distance from the center as [point] A are reflected to [point] A from point E, which is what was proposed.

[2.396] From these considerations it is clear that, if [center of sight] A is taken outside the circle—let it be at H [figure 5.2.41a, p. 262]—and if diameter HDBG is drawn along with tangents HT and HQ, then reflection of points that do not lie the same distance from the center as H can take place to H from any point on arc TG other than from T or G. And the proof will be the same [as the previous one].¹¹⁸

[2.397] **[PROPOSITION 42]** On the basis of these determinations, it will be established that, if reflection occurs to [point] A from point E or from another point not lying the same distance as point A from the center, the diameter in which the point [whose form is] reflected lies forms two angles with diameter ABG, one of them opposite the reflected angle, the other adjacent to it, and that adjacent angle will sometimes be greater than the reflected angle and sometimes smaller.¹¹⁹

[2.398] For instance, draw FB [figure 5.2.42, p. 262] perpendicular to EO. BA is either perpendicular to it or not.

[2.399] Let it be perpendicular. Accordingly, EA will be parallel to FB, and the two angles FBA and FEA will sum up to two right angles. Now, when line BO is drawn, the two angles OBA and OEA will sum up to less than two right angles. Hence, angle OBG > angle OEA, which is the reflected angle. And since triangle EBF = triangle EBA, BF = BA, and so OB > BA.

[2.400] Moreover, when line BN is drawn, the two angles NBA and NEA will sum up to more than two right angles. Thus, angle NBG < angle NEA, and NB > BA, and so [the forms of points] N and O will be reflected to [point] A from point E. Also, they lie at different distances than point A from the center, and diameter OB forms an angle with ABG that is greater on the side of G than the reflected angle [OEA], while diameter NB is longer [than diameter AB], so [we have demonstrated] what was proposed.

[2.401] If, however, BA is not perpendicular to EA, then draw the perpendicular BK, which either lies above AB [figure 5.2.42a, p. 262], or lies below it [figure 5.2.42b, p. 262]. The proof will be the same [in both cases].

[2.402] Let BF be perpendicular to EO, draw FT equal to AK, and draw TB. It is evident that in triangle KEB right angle EKB = [right angle] EFB, and angle KEB = angle FEB [by construction]. It follows that the third [angle, i.e., EBK] = the third [angle, i.e., EBF], and since side EB is common to both triangles [EBF and EBK], the triangles will be equal, and FB = KB. But AK = FT [by construction. And since triangle BAK = triangle BTF] AB = BT, and angle ABK = angle FBT.

[2.403] With common angle [KBT] added to [both in order to yield angle] FBA, [angle] KBF = [angle] TBA.¹²⁰ But [angles] KBF and FEA sum up to two right angles,¹²¹ so [angles] TBA and TEA sum up to two right angles, and so [angle] TBG = angle TEA, which is the reflected angle.

[2.404] Therefore, if a line is drawn from point B to [fall on] line ET beyond T, it will form an angle with BG on the side of G that is smaller than the reflected angle. And that line will be longer than AB, because TB = AB.

[2.405] Furthermore, any line drawn from point B to [fall on] line ET in front of T [i.e., between T and E] will form an angle with BG that is greater on the side of G than the reflected angle, and it will be unequal to AB, so [we have demonstrated] what was proposed.¹²²

[2.406] **[PROPOSITION 43]** Now, let B be the center of sight and G the center of the sphere. Draw diameter ZBGD [figure 5.2.43, p. 263], and produce the plane that contains the diameter and cuts the sphere along [great] circle ZEH. I say that, if point A is reflected to point B from some point on the circle, and if the distance from point A to the center is not the same as that of point B from the center, diameter AG will form an angle [AGD] with

diameter GD on the side of D that cannot possibly be equal to the reflected angle.

[2.407] Let it be equal, let T be the point of reflection, and let $AG \neq BG$. Draw lines TA, TG, and TB, and construct a circle passing through the three points A, G, and B, and this circle will necessarily pass through point T. For if the circle lies beyond it, and if lines are drawn from points A and B to the corresponding point on that outlying circle, they will form an angle smaller than angle ATB. And it will be proven that it is equal.

[2.408] For [angle AGD] along with angle AGB will sum up to two right angles, and angle ATB along with angle AGB sums up to two right angles, since angle ATB = angle AGD, by construction [and within the circle angle ATB + angle AGB = two right angles by Euclid, III.22], and so it is impossible [for T to lie outside the circle]. Likewise, if the circle were to pass below T, the same disproof would hold [insofar as the new angle ATB would be greater than ATB within the circle].

[2.409] It follows from this that it must pass through point T, but since angle ATG = angle BTG [by supposition], arc AG = arc BG, and so [chord] AG = [chord] BG. Yet they have been assumed to be unequal, and so [we have demonstrated] what was proposed [i.e., that angle ATB cannot equal angle AGD when $AG \neq BG$].¹²³

[2.410] **[PROPOSITION 44]** Furthermore, if the two points A and B [figure 5.2.44, p. 264] are taken on the two diameters EGH and ZGD so that $BG > AG$, I say that, if [the form of] point A is reflected to [point] B from two points on arc EZ, the angles of reflection will not both be smaller than angle AGD.

[2.411] For, on arc EZ select two points, T and Q, from which [the form of point] A is reflected to [point] B, and draw lines BT, GT, AT, BQ, GQ, and AQ. If angle ATB < angle AGD, I say that angle AQB will not be smaller than angle AGD.

[2.412] Let it in fact be smaller, draw line GN bisecting the angle formed by the diameters [i.e., angle BGA], and draw line AB, which GN intersects at point F. It is evident that $BG:GA = BF:FA$ [by Euclid, VI.3]. But $BG > GA$; [so] $BF > FA$.

[2.413] Bisect AB at point K, and construct a circle passing through the three points A, B, and T, a circle that will not pass through G, because [if it did] angles AGB and BTA [in quadrilateral BGTA] would be equal to two right angles [by Euclid, III.22], and it is obvious that they are less [than two right angles], since angle BTA < angle AGD [by supposition, and angle AGD + adjacent angle AGB = two right angles]. It will therefore pass above G.

[2.414] By the same token, it will not pass through Q. For if the point, i.e., M, is taken on the circle where line GQ intersects it, arc AM would be equal to arc MB, since [by Euclid, III.26] they would subtend equal angles [MQA and MQB] at Q, which is impossible, because, if point O, where line GT intersects this circle, is taken, arc AO = arc OB, since they subtend equal angles at T [and so GM, which supposedly bisects circle BMA, would cut a smaller arc on the side of A than on the side of B]. It follows that this circle passes above Q, for if it passed below, the same disproof would apply.

[2.415] Now, draw a line from point O to point K [which is the midpoint of BA, by construction], and since it bisects chord AB and likewise bisects arc AB, this line will be perpendicular to AB. But [by Euclid, I.18] angle BAG > angle ABG, since BG > GA. Also, [exterior] angle BFG [of triangle GFA] equals the sum of the two [interior] angles FAG and FGA [by Euclid, I.32], while [exterior] angle AFG [of triangle BFG] equals the sum of the two [interior] angles FBG and FGB [by Euclid, I.32].

[2.416] But [angle] AGF = [angle] FGB [by construction], and [angle] FAG > [angle] FBG [because it is subtended by BG, which is longer than AG]. Thus [for the same reason], angle BFG > angle AFG. Accordingly, [angle] AFG is less than a right angle, so [vertical angle] NFB is less than a right angle. But OK forms a right angle on FB. Therefore, when it is extended, it will intersect GN above BF, never below it.

[2.417] Now, if a circle is produced to pass through the three points A, Q, and B [figure 5.2.44a, p. 265], it will pass above G, and GQ will bisect its arc A[Q]B [since angles AQG and BQG are presumed equal]. But K bisects chord AB [by construction]. Hence, KO will intersect GN below BF and above point G. Therefore, it will intersect GN sooner [than it should, i.e.,] below FB, and it has already been disproven [that it can intersect below FB].

[2.418] It therefore follows that angle AQB may not be smaller than angle AGD, or else [the form of point] A is not reflected to B from point Q. The same disproof will hold if any [other] point is taken on arc EN.¹²⁴

[2.419] Now, if [some] point C is taken on arc NZ [figure 5.2.44c, p. 267], and if reflection of [the form of] point A to B occurs from point C so that the reflected angle at C is less than angle AGD, just as the reflected angle at T is less than the same angle, [the supposition that the reflected angle at C can be less than angle AGD] is disproved in the following way.

[2.420] Draw AC, BC, and GC. It follows necessarily that GC intersects KO on arc A[O]B, and line GC will bisect that arc on circle ABT, as will line KO. Accordingly, let point L be where lines GC and KO intersect. When line TC is drawn, then, since the two lines GC and GT are equal [being radii of circle ZHDE], the two angles GCT and GTC will be equal, and both of them will be acute.

[2.421] Hence, if the line [TX] perpendicular to GT at point T is drawn, it will be tangent to the circle on the mirror, and when it is extended it will fall upon the endpoint of the smaller circle's diameter [by Euclid, III.31], since the angle it forms with TG is subtended [by an arc equal to] a semicircle on the smaller [circle]. Since TO falls upon KO, and since KO, when extended, passes through the center of the smaller circle [by virtue of its bisecting line AB along the orthogonal], that perpendicular [TX] will necessarily fall upon the endpoint of KO, when it is extended [by Euclid, III.31], and TC lies below that perpendicular when it is taken with respect to N.

[2.422] Therefore, no matter what line is drawn [from G] to line TC so as to intersect diameter OK of that circle, it will fall to a point on line TC but below that perpendicular [TX]. Therefore, since GC falls to C and intersects OK, C will lie below the perpendicular and beneath the arc [on circle AOBT passing] through that perpendicular.

[2.423] Therefore, if a circle is produced to pass through the three points A, B, and C, it will pass through C, and it will intersect circle ABT at the two points A and B. And when it continues past point B, it will pass on to point C, and although it lies below that circle [i.e., ABT], it will necessarily intersect it at a third point, which is impossible.¹²⁵

[2.424] It follows, then, that [the form of point] A may not be reflected to B from two points on the arc extending between the diameters on which they lie, i.e., arc EZ, in such a way that both angles of reflection are less than angle AGD, which is what was proposed.

[2.425] **[PROPOSITION 45]** I say, moreover, that two points lying different distances from the center can be reflected from two points on the arc facing them, that is, the arc lying between the diameters in which those points lie.

[2.426] For instance, taking two diameters, i.e., BD and GD [figure 5.2.45, p. 268] in the [great] circle of the sphere, let the angle formed by them be bisected by diameter ED, and take point M on BD beyond the point where the perpendicular dropped from point E to BD will fall. Then take ND equal to MD, and construct a circle passing through the three points D, N, and M. That circle will necessarily pass beyond E, for if [it passed] through E, it would create a quadrilateral at the four points D, N, E, and M, and the two opposite angles of that quadrilateral are equal to two right angles, which would not be so [in this case], since line EM lies beyond the perpendicular, and angle EMD is acute.

[2.427] Likewise, its counterpart at N is acute, because EN [also lies] beyond the perpendicular [so EMD and END sum up to less than two right angles]. The disproof will be similar if the circle is [assumed] to pass below

E. Therefore, it will pass beyond, and it will intersect the [great] circle of the sphere at two points, e.g., T and L.

[2.428] Draw lines MT, DT, NT, ML, DL, and NL, and draw line MN to intersect [line] TD at point F and line ED at point P. It is evident that, since $MD = ND$ [by construction], since PD is common, and since the [subtended] angle [MDP] = the [subtended] angle [NDP, by construction], triangle [MDP] = triangle [NDP], and angle FPD will be a right angle. Hence, angle PFD is acute.

[2.429] From point F draw KF perpendicular to TD. It is clear that any point on line NL will lie below point K [since FK intersects ML between M and L]. Taking the position of [some such] point [on NL] below [K] with respect to T, let that point be Z, and draw line TZ until it reaches point C on the circle. Arc NC is either shorter than arc TL or not.

[2.430] If it is not shorter, then take an arc on it [i.e., on arc NC] that is shorter, and draw a line from point T to the endpoint of that arc, and it will be the very line [TC that we wanted in the first place].

[2.431] So let $NC < TL$. It is obvious that angle $TNL >$ angle CTN , since it is subtended by a longer arc [i.e., TL as opposed to CN]. Cut an [angle] equal [to CTN] from it, let that angle be INZ , and at point T form angle OTM equal to angle CTN . Since angle $TML >$ angle MTO ,¹²⁶ line TO will intersect line LM. Let it intersect at point O.

[2.432] Hence, since angle LMT [exterior to triangle MOT] = [the sum of] the two [interior] angles MOT and MTO [by Euclid, I.32], since angle $LNT =$ angle LMT , because they are subtended by the same arc [TL], and since angle $INZ =$ angle MTO [by construction], angle INT [which = $LMT - INZ$] = angle MOT [which = $LMT - MTO$], and so triangle MOT is similar to triangle INT, and likewise triangle INZ is similar to triangle TNZ . So $NT:TO = NI:MO$, and likewise $TN:TZ = IN:NZ$.

[2.433] But $TZ > TO$, which becomes clear as follows. Let R be the point where TZ intersects KF. Angle TFR is a right angle [by construction], so angle FTR is acute. Hence, angle OTF , which equals it, is acute.¹²⁷ Moreover, KF is perpendicular to TD [by construction], so when it is extended it will intersect TO, and the line [TX] drawn from point T to that point of intersection, a line that includes TO as a segment, will be equal to line TR [because triangles TRF and TXF are equal, by Euclid, I.26]. And thus $TO < TZ$, so $NT:TO > NT:TZ$.

[2.434] Therefore [since we have already established that $NT:TO = IN:MO$ and that $NT:TZ = IN:NZ$], $IN:MO > IN:NZ$, so $MO < NZ$. Accordingly, from NZ cut off segment NS equal to MO.

[2.435] Since angle $LND +$ angle $LMD =$ two right angles [by Euclid, III.22], angle $LND =$ angle OMD [which is adjacent to LMD], and $SN + ND$

= OM + MD [since MD = ND, and SN = OM, by construction]. Hence OD = SD [because triangles SND and OMD are equal, by Euclid, I.4].

[2.436] But $ZD > SD$, because angle LND + angle LMD = two right angles [by Euclid, III.22]. Angle LMD is acute, however, since angle EMD is acute. Therefore, angle LND > a right angle. Hence, $ZD > SD$ [because it intersects LN farther from vertex N than does SD], so $ZD > OD$ [because $SD = OD$, by previous conclusions].

[2.437] Accordingly, [the form of point] O is reflected to Z from the two points T and L [given that $OTD = ZTD$, by previous conclusions, and $ZLD = OLD$, since they are subtended by equal arcs], and O and Z lie unequal distances from the center, and they lie on different diameters.

[2.438] That they do not lie on the same diameter is evident from the fact that angle SDN = angle ODM. Thus, when the common angle SDM is added [to each] angle, [angle] NDM = angle SDO. But angle NDM < two right angles, so angle ZDO < two right angles [which means that Z, D, and O cannot lie on a single straight line]. Therefore, O and Z do not lie on the same diameter but on different ones.

[2.439] **[PROPOSITION 46]** Furthermore, given two points O and K [figure 5.2.46, p. 269] that lie different distances from the center [of the mirror], one will be reflected to the other from two points on the arc facing the radii in which those point lie, but they will not reflect from any point on that arc other than those two.¹²⁸

[2.440] For example, let D be the center [of a great circle on the mirror], let K lie farther from D than O does from D, let GD and OD be diameters, and let T be one point of reflection. It is clear from earlier discussions [in propositions 43 and 44 above] that the two reflected angles will not [both] be smaller than, nor equal to, angle ODA. Hence, one of them will be greater. Let the reflected angle at point T be greater [than angle ODA], and draw lines OT, DT, and KT.

[2.441] Then, from that reflected angle [OTK] cut off [angle] OTF equal to angle ODA, and bisect angle FTK with line TE. From point K draw [line KZ] parallel to TF, a line that will intersect TE. Let it intersect at point Z, draw line OK, bisect angle ODK with line DU so that it intersects line OK at point C, and let $KD > OD$. Therefore, since $KD:DO = KC:CO$ [by Euclid, VI.3], $KC > CO$. Then, let line DT intersect line OK at point N. I say that C lies between N and K, not N and O, which will be shown as follows.

[2.442] Angle KCD [exterior to triangle CDO] = the two [interior] angles CDO + COD [by Euclid, I.32], and angle OCD [exterior to triangle CDK] = the two [interior] angles CKD + CDK. But angle CDO = angle CDK [by construction], and [by Euclid, I.19] angle KOD > angle OKD [since $KD >$

OD]. Hence, angle KCD > angle OCD, so angle KCD > a right angle. Also, angle KND is acute, which will be established thus.

[2.443] If a circle is constructed through the three points O, T, and K, it will pass below D, because, since angle OTK > angle ODA [by construction], the two angles OTK + ODK > two right angles, and line ND will bisect arc OK of that circle below D.¹²⁹

[2.444] If a line is drawn from the point of bisection [s] to the midpoint [x] of line OK, which is the chord on that arc, that line will be perpendicular to OK [by Euclid, III.3], and it will fall between C and K, since CK > CO [by previous conclusions]. Moreover, the angle at [point] N beyond that perpendicular on the side of C [i.e., DNC] will be acute, and the angle at C on the side of O [i.e., OCD] is acute [since angle KCD is obtuse, by previous conclusions]. Accordingly, if C were to fall between N and O, it would be impossible for that perpendicular to fall between N and C, because it would intersect DC and would form a triangle with one angle right and the other obtuse.¹³⁰

[2.445] Hence, it [i.e., the perpendicular] will fall between N and K, and the angle at N on the side of the perpendicular will be acute, so this [same angle] will be obtuse on the side of C [if C lies between N and O], and so there will be a triangle with two obtuse angles.¹³¹

[2.446] It is evident that angle KTD is half of angle KTO [by construction], but [angle] KTE is half of angle KTF [by construction]. It follows that [angle] ETD is half of angle FTO [since ETK = FTK, by construction], but [angle] FTO = angle ODA [by construction]. Thus, [angle] ETD is half of angle ODA.

[2.447] But angle ODA + [adjacent] angle ODF = two right angles, and the three angles of triangle ETD sum up to two right angles. With common [angle] EDT subtracted, [angle] TED remains equal to half of angle ODA + angle ODN. But angle ODC + half of angle ODA is a right angle [since 2ODC (which = ODK) + ODA = two right angles]. Therefore, angle TED is acute [since TED = ODC – NDC + half of ODA], so its vertical angle [KEZ] is acute.

[2.448] Thus, if a perpendicular is dropped from point K to TZ, it will fall between E and Z. For if it were to fall above E, then, since angle TEK is obtuse, it would follow that the triangle [formed by KD and the perpendicular intersecting TE above E] would have two [of its three] angles [consisting of] a right angle [formed by the line from K that intersects TE above E] and an obtuse angle [KET]. Let KQ be the perpendicular, then. I say that KT:TF = KD:DO.

[2.449] The proof is as follows. Either TO is parallel to KD, or it intersects it. Let it be parallel [figures 5.2.46a and 5.2.46b, p. 270]. Angle ODA =

[alternate] angle TOD, and so [angle] TOD = angle OTF [which = angle ODA, by construction]. Either OD and TF are parallel, or they will intersect.

[2.450] If they are parallel [figure 5.2.46a, p. 270], then, since they fall between parallels, they will be equal. If, however, they intersect [figure 5.2.46b, p. 270], then they form a triangle [two of] whose sides [OP and TP] are equal, because they subtend equal angles [since angle DOT = alternate angle ODA, and angle OTF = angle ODA, by construction], and because FD intersects those sides parallel to the base [TO]. Therefore, the ratio of one side to DO will be the same as the ratio of the other side to TF, and so $TF = DO$.

[2.451] [I claim that this is so if they [i.e., DO and TF] intersect below KD. And if they intersect below TO, the same proof will apply, because they will form a triangle one of whose sides is TO and the other two of which are equal, and the ratio of one [of those equal sides] to DO will be the same as the ratio of the other to TF.]¹³² Furthermore, angle TDK = [alternate] angle DTO, because DT lies between parallels. Thus, it [i.e., angle TDK] = angle DTK [which = angle of reflection DTO], so DK and TK are equal. Hence, $TK:TF = KD:DO$.

[2.452] If, on the other hand, TO intersects KD, let it intersect at point P on the side of A [figure 5.2.46c, p. 270]. We know that $KT:TF$ is compounded of $KT:TP$ and $TP:TF$ [i.e., $KT:TF = (KT:TP):(TP:TF)$]. But $KT:TP = KD:DP$ [by Euclid, VI.3], because DT bisects angle KTO. Moreover [given that triangles TPF and DPO are similar], $TP:TF = DP:DO$, because angle ODP [i.e., ODA] = angle PTF [by construction], and the angle at P is common. Part [ODP] of the [larger] triangle [PTF] is [therefore] similar to the whole. Hence, $KT:TF$ is compounded of $KD:DP$ [which = $KT:TP$] and $DP:DO$ [which = $TP:TF$]. But $KD:DO$ is compounded of the same [ratios, i.e., $KD:DP$ and $DP:DO$], so $KT:TF = KD:DO$.

[2.453] If, however, TO intersects KD on the side of G [figure 5.2.46d, p. 270], let L be the [point of] intersection, and from point D draw [line] DR parallel to line KT so as to intersect TO at point R. Accordingly, angle KTD = [alternate] angle TDR, but it is [also] equal to angle DTO [by supposition], so $DR = TR$. However, since triangle LTK is similar to triangle LRD [because TK and RD are parallel], $DR:RL = KT:TL$, and so RT [which = DR]: $RL = KT:TL$. But $RT:RL = DK:DL$. Thus, $KT:TL = KD:DL$.

[2.454] Yet, since angle FTO = angle ODA [by construction], angle ODL [adjacent to ODA] = angle FTL [adjacent to FTO], and the angle at L is common. [Thus], triangle ODL will be similar to triangle FTL. Hence, $TL:TF = DL:DO$, and so [by previous conclusions] $KT:TL = KD:DL$. Moreover [by previous conclusions], $TL:TF = DL:DO$, so $KT:TF = KD:DO$, which is what was proposed.¹³³

[2.455] However, since KZ [in figure 5.2.46, p. 269] is parallel to TF [by construction], angle KZE = [alternate] angle ETF [and vertical angle ZEK = vertical angle TEF], so triangle KZE is similar to triangle ETF, and so $KE:EF = KZ:TF$. But [by Euclid, VI.3] $KE:EF = KT:TF$, because the angle at T is bisected. Thus, $KZ = KT$.

[2.456] Since KQ is perpendicular to EZ, though, all of its angles [of intersection with TZ] will be right angles. But angle ETD is acute, because it is half of angle [FTO, by earlier conclusions]. Therefore, KQ will intersect TD. Let H be the intersection, draw line EH, and from point E draw EC' parallel to KH, and extend it to DH [which it intersects at point C'].

[2.457] Adjust the figure according to the interrelationship of lines [figure 5.2.46e, p. 271], and construct a circle that passes through the three points C', T, and E. Extend KD until it reaches the circle at point M, and draw MT. Angle TME = angle TC'E, since they are subtended by the same arc, and angle TC'E = angle C'HK [since EC' and KH are parallel, by construction]. [Therefore, angle] TME = angle C'HK.

[2.458] Cut from angle TME angle F'MD equal to angle DHE, and let I be the point where F'M intersects TC'. It is evident that triangle IMD is similar to triangle EHD [because all their corresponding angles are equal], so $HD:DM = EH:IM$.

[2.459] Likewise, triangle TMD is similar to triangle KHD [because angle TMD (i.e., TME) = angle TC'E (which = angle THK), by construction, and angle HDK = vertical angle TDM], and [therefore, in similar triangles TMD and KHD, the corresponding sides are proportional, so] $KD:DT = HD:DM$, and so [given that $HD:DM = EH:IM$, by previous conclusions] $KD:DT = EH:IM$.

[2.460] But $KD:DT$ is known, since it remains one and the same throughout, no matter where point of reflection T might lie on arc EG, because TD remains unchanged, and so does KD. Line EH also remains one [and the same] no matter the reflection, so it does not change its length, and so line IM will always be one [and the same], and so point F' is known and determinate.¹³⁴

[2.461] Therefore, if reflection could occur from three points on arc BG, three lines equivalent [to FM] could be extended from point F' to circle TC'E, because $KD:DT$ would be as EH is to any one [segment IM] of them. But it is clear from earlier discussion [in proposition 20, lemma 2 above] that only two equal lines can be [so] drawn, so reflection will occur from only two points, which is what was proposed.¹³⁵

[2.462] **[PROPOSITION 47]** Furthermore, given two points K and O [figure 5.2.47, p. 275] lying on different diameters and at different distances from the center, to find [either] point of reflection.

[2.463] For example, take line ZT' , and cut it at point E so that $ZE:ET' = KD:DO$. Since $KD > DO$ [by construction from the previous proposition], $ZE > ET'$. Bisect ZT' at point Q , from point Q draw a line $[K'QH]$ perpendicular to ZT' , and form angle $ET'D'$ equal to half of angle ODA . It will be acute. Accordingly, $T'D'$ will intersect the perpendicular $[KQ]$.

[2.464] Let the intersection be at point H , and [by proposition 24, lemma 6] draw line $D'EK'$ so that $K'D':D'T' = KD:[DT]$, the radius of the sphere. Then, form angle KDT in the mirror equal to angle $K'D'T'$ that we have [in the construction based on $T'Z$]. I say that T is a point of reflection, and if you repeat the previous proof, you will see this clearly.¹³⁶

[2.465] **[PROPOSITION 48]** Moreover, if two points are taken on different diameters, if they lie at different distances from the center, and if they lie outside the circle and are reflected from some point on the [concave] arc facing the diameters, they will not be reflected from any other [point] on the same arc.

[2.466] For example, let A and B [figure 5.2.48, p. 276] be points lying outside the circle on different diameters, G the center [of the great circle on the mirror], and T the point of reflection, and draw BT , AT , and GT . BT will intersect the [convex] arc on the circle. Let Q be the point of intersection. So too, AT will intersect the [convex] arc on the circle. Let M be the point of intersection.

[2.467] Since angle $BTG =$ angle ATG [by construction], they are subtended by equal arcs on the circle, which will be evident if diameter TG is drawn. Accordingly, arc $QT =$ arc MT [and so, therefore, do chords QT and MT]. Thus, if [the form of point] B is reflected to [point] A from a point other [than T], let it be H , and draw lines BH , AH , and GH . Let BH intersect the circle at point L , AH at point N .

[2.468] According to the previous reasoning, $HL = NH$ [because they subtend supposedly equal arcs]. But we have just [established] that $QT = TM$, which is impossible [if $HL = NH$]. It follows that [the form of point] B may not reflect to [point] A from point H or from any point other than T on the arc facing [either] diameter.

[2.469] Likewise, if one of the points lies on the circle, while the other lies outside, then the one can be reflected to the other from only one point on the arc.

[2.470] Moreover, if the line extending from one of the two points [A or A' in figure 5.2.48a, p. 276] to the other [B] is tangent to the circle or lies entirely outside it, then, if some point [T] is taken on the [convex] arc facing the diameters, one of the lines [i.e., either AT or $A'T$] extended from [each of] the two points to that point will lie entirely outside the circle [because it

will strike its outer rather than its inner surface]. And so neither of the points [A and B, or A' and B] will be reflected to the other from any point on that arc [CE], and [it will be reflected] from only one point on the opposite arc [KD] of the mirror, and so [it will be reflected] from only one point on the [entire] mirror.

[2.471] **[PROPOSITION 49]** On the other hand, if the line extending from one point to the other [i.e., from object-point to center of sight] cuts the [great] circle [on the mirror], construct the circle [passing] through the center of the mirror and those two points. That [second] circle will lie entirely within the circle [of the mirror], or it will touch it [at one point], or it will intersect it.

[CASE 1]

[2.472] Let it lie entirely within [figure 5.2.49, p. 277], and draw two lines [AT and BT] from the two points to some point [T] on the facing arc [of the mirror]. The angle they will form [ATB] will be smaller than the angle one diameter forms with the other on the adjacent side of the [mirror's] center [i.e., angle DGB]; and no matter what angle is formed in this way on the facing arc, it will be smaller than the latter angle.

[2.473] For the angle [ACB] formed within the inner circle by the lines drawn from the points to the arc on it lying between [the two points] will be equal to that latter angle [DGB], because, combined with the angle [AGB] formed by the diameters above the center, it sums up to two right angles [by Euclid, III.22]. But the angle [ACB] within arc [ACB] in the smaller circle is greater than the angle [ATB] within arc [CTH of the circle] on the mirror.

[2.474] Therefore, in the arc of the circle [on the mirror], reflection will occur from only one point, since it has already been claimed [in proposition 44 above] that it is not possible for reflection to occur from two points such that both [reflected] angles are smaller than the angle formed by the diameters on the adjacent side of the center.

[CASE 2]

[2.475] If, however, that [second] circle touches the circle of the mirror at one point [T in figure 5.2.49a, p. 277], the angle formed by the lines drawn from those points [A and B] to the point of contact [T] will be equal to the angle [DGB] formed by the diameters on the adjacent side of the center [because $AGB + DGB =$ two right angles, by Euclid, III.22, and so do $AGB + ATB$], so no reflection will occur from that point of contact [according to proposition 43 above]. Moreover, the angle formed at any other point [e.g., T'] on the arc [CH] of the larger circle will be smaller than that one, so,

according to previous claims [in proposition 44 above], reflection will not occur from two points on that arc.

[CASE 3]

[2.476] If, on the other hand, the inner circle cuts the circle of the mirror, the two points [A and B] will lie outside the circle [of the mirror], or [they will both lie] inside it, or one [will lie] inside and the other outside, or one [will lie] on the circle and the other outside or inside it.

[SUBCASE 3A]

[2.477] If they [both] lie outside [as represented by A and B in figure 5.2.49b, p. 277], or if one lies on the circle and the other outside [as represented by A and B' in the same figure], then the [second] intersecting circle will not cut the arc of the mirror's circle between the diameters, and so, no matter what angle is formed on that arc [e.g., ATB or ATB'], it will be larger than the angle [DGB] formed by the diameters on the adjacent side of the center. And it has already been demonstrated in the preceding figure [i.e., proposition 48] that [the forms of] these points can be reflected from only one point on the [concave] arc [KD] lying between [the diameters].¹³⁷

[SUBCASE 3B]

[2.478] But if the two points lie inside [the mirror's circle, as represented in figure 5.2.49c, p. 277], the inner circle will cut the arc lying between [the diameters] at two points [E and F], and there will be two arcs [CE and FH] left over on opposite sides [of arc CH facing the diameters].

[2.479] If one of the points lies inside the circle and the other on the circle or outside it [as represented in figure 5.2.49d, p. 278], the [second] circle will cut the arc [CB'] lying between [the diameters] at a single point [E], and only one arc-[segment, CE] will be left over.

[2.480] If it intersects [the arc between the diameters] at two points, all the angles [e.g., ATB in figure 5.2.49c, p. 277] formed upon the arc lying between the two points of intersection [E and F] will be greater than the angle [DGB] formed by the diameters on the adjacent side of the center, and from this arc [EF] reflection may occur from only one point, or it may occur from two [by proposition 46 above].

[2.481] And from the two arcs [CE and FH] that are left over from the entire arc on opposite sides of [points E and F of intersection], all the angles [e.g., AT'B] will be smaller than the angle [DGB] formed by the diameters [on the adjacent side of the center], and reflection will occur from only one point on them [by proposition 44 above].

[2.482] And [so] in this case [when two arc-segments are left over on opposite sides of intersection-points E and F], reflection can occur from two

points on the arc lying between the diameters, or from three points [in the whole arc CH, i.e., two from EF and one from CE or FH].

[2.483] Moreover, it is clear [from proposition 38 above] that reflection will occur from only one point on the opposite arc [DK], so in this case it may occur from three [points], or it may occur from four.¹³⁸

[2.484] But if it [i.e., the second circle] cuts the arc lying between the diameters at only one point on the larger circle, all the angles formed on the segment of that arc [EB' in figure 5.2.49d, p. 278] included within the smaller circle will be greater than the angle [DGB'] formed by the diameters [on the adjacent side of the center], and [so] reflection can occur from two points, or from one point, on that part [CB' of the arc facing the diameters].

[2.485] All the angles in the other side [CE] of the arc lying between [C and B'] will be smaller than the angle formed by the diameters [on the adjacent side of the center], and reflection will occur from only one point on that segment, and so, given that reflection always occurs in this case from one point on the opposite arc [DK of the mirror], reflection may occur from three [points], or it may occur from four, but never from more.

[2.486] It is therefore clear that points lying different distances from the center may at times be reflected from only one point, at times from two, at times from three, or at times from four, but never from more [than four]. Moreover, if the [two] points lie the same distance [from the center], reflection can occur from one point only, or from two, or from four, [but] never from three [alone].¹³⁹

[2.487] When reflection occurs from one point, one image appears; when [it occurs] from two, two [images appear]; when [it occurs] from three, three [images appear]; when [it occurs] from four, four [images appear]. But if the visible point and the center of sight lie on the same diameter, reflection will occur from an entire circle [within the sphere of the mirror], and the image-location will be [at] the center of sight [by proposition 36 above]. If the center of sight lies at the center of the mirror, though, it sees nothing [other than itself]. On the other hand, if the visible point lies at the center of the mirror, it[s image] will not be seen, because its form will reach the mirror along the normal and can be reflected only along the normal.

[2.488] Yet when the center of sight and the visible point lie outside the center on different lines, those lines, when extended to the center, will cut two arcs on different sides of the circle within the sphere. Reflection will occur from only one point on one [of those arcs], but [it may occur] from three [points] on the other. But if the center of the sphere lies on one side, while the center of sight and the visible point lie on the other, the arc that the diameters cut [on one side or the other in the circle] will be blocked by the [viewer's] head, so reflection will then occur from only three points [at

most]. And if in that case the eye is directed to the arc where only a single reflection occurs, the other arc [from which] three [reflections occur] will be blocked, and only one image will appear.

[2.489] Furthermore, if the [surface of the] mirror is fully closed, no image will be perceived in it. Hence, there must be some opening in it, and it will sometimes happen that the arc lying between the diameters is open, in which case nothing can be seen in it, so it will rarely turn out that four images are seen in this [sort of] mirror. Hence, if one wants to see this multiplicity of images, he should position his eye inside the mirror [but] near its surface so that only a small part of it is blocked by the mass of his head and so that he can scan the entire surface of the mirror with his eye.

[2.490] When something is perceived in this [sort of] mirror with both eyes, if the line of reflection is parallel to the normal, the image-location will lie at the point of reflection [by proposition 32 above], and since the points of reflection are separated from one another with respect to the two eyes, two images of the same point will appear to the two eyes [and will be melded at the point of reflection]. On the other hand, if the line of reflection is not parallel to the normal, and if the visible point lies the same distance from one eye as from the other, or if the difference [in distance] is slight, the image-location will be the same for both eyes, or it will be different [for each eye] but only slightly divergent. Therefore, either a single image, or virtually a single image, will appear, as was demonstrated in the case of convex spherical mirrors [in paragraph 2.221 above].

[Concave Cylindrical Mirrors]

[2.491] In the case of concave cylindrical mirrors, the common section [of the plane of reflection and the mirror] is sometimes a straight line. When the plane of reflection intersects the axis, the common section is sometimes a circle—[i.e.,] when that plane is parallel to the bases [of the cylinder]—[and] the common section is sometimes a cylindric section [i.e., an ellipse]. When [the common section] is a straight line, image-location and the analysis of reflection will be the same as in plane mirrors. When it is a circle, the analysis will be the same as in concave spherical [mirrors]. However, when the [common] section is cylindric [i.e., elliptical], the image-location will lie behind the mirror, or beyond the center of sight, or at the center of sight, or between the mirror and the center of sight, or on the mirror itself, which will be demonstrated in the following way.

[2.492] **[PROPOSITION 50]** Let ABG [figure 5.2.50, p. 280] be the [elliptical common] section. Draw normal DG within this section, and, ac-

cording to previous discussion [in book 4], it is clear that this normal is a diameter of the circle [parallel to the cylinder's base and coincident with the section], and it must be unique, because from no other point on the section can a normal be drawn to the plane tangent [to both the circle and the section].¹⁴⁰ Select another point [on the section], let it be B, and from it draw a line within the section that is normal to the line tangent to the section at point B, and, as claimed earlier [in proposition 26], this line will necessarily intersect normal [GD]. Let it intersect at point D, and let [point] B be chosen near enough to point G that angle BDG is acute.

[2.493] Then, from point G, draw line GH parallel to BD within the section, and it should lie within the cylindric section, because angle HGD will be acute, since it is equal to [alternate angle] GDB. From point G draw a line between D and H, and it will necessarily intersect BD. Let it intersect at point N, and between N and G select some point O. Beyond point N [on line GN] select point T. Furthermore, from point G draw another line GZ above GH, [but] still within the section, and it will necessarily intersect BD on the other side [of G]. Let E be the [point of] intersection. Draw line GQ so that angle QGD = angle ZGD, and form angle LGD equal to angle HGD and angle MGD equal to angle NGD.

[2.494] It is evident that, if the center of sight lies at point Z, [the form of] point Q will be reflected to it from point G [since QGD = ZGD by construction], and point E [behind the mirror on normal QD will be the location] of its image. If the center of sight lies at point H, [the form of] point L will be reflected to it from point G [since LGD = HGD by construction], and its image-location will be [point] G [on the mirror's surface, because normal LD is parallel to line of reflection GH]. If the center of sight is at point O, [the form of] point M will be reflected to it [from point G, since MGD = NGD by construction], and its image-location will be [point] N [behind the eye on normal MD]. If [the center of sight] lies at N, the image-location [for the form] of point M will be at the center of sight [itself], i.e., at N [where normal MD intersects line of reflection GN]. And if [the center of sight] is at T, the image-location [for point M] will lie between the eye and the mirror, because it lies at N, and so [we have demonstrated] what was set out [to be proven].

[2.495] These conclusions must be understood [to apply] when the visible point does not lie on the [same] normal as the center of sight, for in that case, since an infinite number of planes can all be imagined to lie on that normal such that each one is orthogonal to the plane tangent to the mirror, and since all of them lie on the normal, any one of those planes will form a rectilinear common section [with that tangent plane]; and reflection will only occur along the same normal, the center of sight [will constitute] the

image-location, and no point will be seen unless it lies on the surface of the eye.

[2.496] However, one of those planes forms a circular common section [with the mirror], and in that case, when the center of the mirror [D in figure 5.2.50a, p. 281] lies between the visible points [e.g., C] and the eye [E], each of those points can be reflected to the eye from two points [e.g., H and L] on the circle, since lines may be drawn from each of them to form an angle with the plane tangent [to the point of reflection such] that the diameter [HDL] drawn [from that point of reflection] to the center [of the circle] bisects [that angle]. I say this, of course, about points that lie on that normal, and their image-locations lie at the center of sight.¹⁴¹ The other points on that normal [i.e., between D and E in figure 5.2.50a] will not be reflected to the eye except for the point that lies on the surface of the eye, and that one [is reflected] along the normal.

[2.497] On the other hand, when the common section is a cylindric section, the points on the normal cannot be reflected from any [other] points on the section, because the form reaching along the normal must be reflected along the normal, and in [such] a section the normal is unique [as shown in book 4], so reflection will occur only along this normal, and only the point on the surface of the eye [will be so reflected], and the image-location will be at the center of sight.

[2.498] If, however, the center of sight lies at the center of the circle, the portion of the eye that the normals extending from the center of sight to the circle [on the mirror] cut off will be reflected from the corresponding portion of the circle [on the mirror] that the normals cut. Since any line extending from the center of sight to the circle is normal, reflection will occur along the normal, and the image-location will be [at] the center of sight, which is the center of the circle.

[2.499] Now, at point A [figure 5.2.50, p. 280] form acute angle FAG of some kind. It is clear that FA will intersect GZ. Let the intersection be at point Z, and form angle CAG equal to angle FAG. AC will intersect GQ. Let the intersection be at point C. It is evident that [the form of point] C is reflected to [point] Z from point G, and it is also reflected to [point] Z from point A, but not from any other point on the section, because it cannot reflect except from the endpoint of the normal, and there is only one such normal in the section, namely, GA.

[2.500] **[PROPOSITION 51]** Moreover, if two points are taken on the axis of the cylinder, [the form of] one will be reflected to the other from one full circle in the cylinder, and the image-locations will lie on a given circle outside the cylinder.

[2.501] For example, Let EZ [figure 5.2.51, p. 282] be the axis, T and H two points selected on the axis, and AG and BD the bases of the cylinder. Bisect TH at point Q, and construct a circle with Q as its center, i.e., LM, that will be parallel to the [cylinder's] bases, LM being its diameter, and BLA and DMG being sides of the cylinder. Also, construct circle KC with H as its center and CK its diameter, and draw lines TL, TM, HL, and HM.

[2.502] It is evident that each of the four angles at Q is a right angle, that $TQ = QH$, and that $QL = QM$. Those triangles [i.e., TLM and MLH] will be similar, so angles TLQ and QLH will be equal; likewise, angles TMQ and QMH will be equal. Therefore, if H is the center of sight, [the form of] point T will be reflected to point H from point L, and likewise from point M. Accordingly, if triangle TLH is rotated while axis TH remains stationary, point L will describe a circle, the two angles TLQ and QLH will remain constant throughout, and throughout this motion [the form of] T will be reflected to H.

[2.503] Now, draw line CHK until it intersects line TL, and let F be the [point of] intersection. It is evident that F will be the image-location, and as triangle TLH revolves, triangle TFH will revolve, and during this motion point F will describe a circle outside the cylinder. That entire circle will be the location for [all] images, and this is what was proposed.¹⁴² The same method of proof will apply for any two points [chosen] on the axis.

[2.504] **[PROPOSITION 52]** Furthermore, some points that are selected outside the normal [extending] from the eye have one image, some [have] two, some [have] three, and some [have] four, but none [has] more [than four].

[2.505] For instance, let A [figure 5.2.52, p. 283] be a visible point outside the normal [extending] from the center of sight, and construct a plane passing through A and parallel to the bases of the mirror. It will, of course, form a circle on the cylinder. Let H be the center of that circle, and choose another point B within the plane of the circle, and draw diameters AH and BH [B thus serving as a center of sight within this plane].

[2.506] From what has been claimed about spherical concave mirrors [in proposition 49 above], it is clear that [the form of point] A may be reflected to [point] B from [at least] one point on the arc [passing through points E, D, and G] that subtends those two diameters, [or] perhaps from two or three, but from no more [than three]; and from the opposite arc [reflection can occur] from only one point. Accordingly, let [the form of point] A be reflected to [point] B from three points on the subtending arc [that passes through points E, D, and G], let those points be G, D, and E, and draw lines AG, HG, BG, AD, HD, BD, AE, HE, and BE.

[2.507] From point A in the same plane draw three lines AK, AF, and AN parallel to the three diameters HG, HD, and HE [respectively]. Hence, since AK is parallel to HG, BG will intersect AK. Let it intersect at point K. Likewise, BD will intersect AF. Let the intersection be at point F. So, too, BE [will intersect] AN. Let the intersection be at point N.

[2.508] Then, from point H erect axis HU, and from point B erect a line perpendicular to the plane of the circle. Let it be BT, and it will be parallel to the axis. Take some point T on it, draw the three lines TK, TF, and TN, and from the three points G, D, and E, erect three lines GM, DL, and EQ [respectively] perpendicular to the plane of the circle [each line thus being a line of longitude on the cylinder's surface]. They will be parallel to TB. Hence, EQ will lie in the plane of triangle TBN. So EQ will intersect TN. Let it intersect at point Q. Let DL intersect TF at point L, and [let] GM intersect TK at point M. These three perpendiculars will constitute lines of longitude on the cylinder.

[2.509] From point Q draw a line parallel to line NA, and it will intersect axis UH, because it will be parallel to EH [which is parallel to NA by construction]. Let the intersection be at point U, and draw line TA, which QU will intersect, because QU is drawn from one side of the triangle [i.e., TNA] parallel to the base [AN, which is parallel to HE by construction]. Let I be the point of intersection, and draw line QA.

[2.510] It is clear that angle BEH = [alternate] angle ENA [since HE and AN are parallel, by construction], while angle HEA = [alternate] angle EAN [for the same reason], and angle [of reflection] BEH = angle [of incidence] HEA [by construction. Consequently] angle EAN = angle ENA, so EN = EA.

[2.511] Moreover, EQ is perpendicular [to the plane of the circle. Accordingly] triangle QEA = triangle QEN [by Euclid, I.4]; [and so] QN = QA, and angle QNA = angle QAN [since triangle QNA is isosceles]. But angle TQI = [alternate] angle QNA [between parallels QI and NA], and angle IQA = [alternate] angle QAN [between parallels QI and NA]. [Hence], angle IQT = angle IQA, so [the form of point] A is reflected to [point] T from point Q on the cylinder.¹⁴³

[2.512] In the same way it will be demonstrated that [the form of point] A is reflected to [point] T from points L and M, and thus from three points on the same side of the cylinder.¹⁴⁴

[2.513] Nor can it be reflected from more points, for let another [such point] be given. If a side [i.e., a line of longitude] is drawn from that point, it will fall on the circle that we have, and, by repeating the proof, it will be demonstrated [according to proposition 49 above] that from the point where the side falls to the circle it is impossible for [the form of point] A to be reflected to [point] T.

[2.514] [But the form of point] A can be reflected to [point] B from one point on the opposite arc of the circle. Let Z [in figure 5.2.52, p. 283] be that point, and draw diameter HZ as well as line AC parallel to it. Then draw BZ, and let it intersect AC at point C. Erect perpendicular OZ, which will be a line of longitude and [thus] parallel to TB, and draw TC, which will be intersected by line OZ. Let the intersection be at point O. It will be proven according to the previous method that [the form of point] A is reflected to [point] T from point O.¹⁴⁵ And if another point from which reflection can [supposedly] occur is chosen on that side of the cylinder, it will be demonstrated by repeating the proof [according to which the line of longitude is dropped from that point to the circle] that it is impossible for [the form of point A] to be reflected [to point B] from any point on that side of the circle other than Z [as demonstrated in proposition 38 above].

[2.515] Therefore, if [the form of point] A is reflected to [point] B from one point on a given side of the circle, it is reflected from one [point] on the same side of the cylinder; if [it is reflected] from two [points on the circle, it will be reflected] from two [points on the cylinder]; if [it is reflected] from three [points on the circle, it will be reflected] from three [points on the cylinder]; but [such reflection] can [occur] from no more [than three points on that side]; whereas on the opposite side [it can occur] from only one point on the circle and [thus] from only one point on the cylinder.

[2.516] Furthermore, TB is parallel to UH, and there can be no plane chosen, other than plane TBUH, in which T lies with UH. Likewise, there can be no plane other than AUH in which A lies with UH, and it is perpendicular [to the plane of the base-circle]. Hence, T does not lie in the same perpendicular plane with A, nor on the same circle [forming its plane], nor is it on the axis, because it lies on a line parallel to it. Accordingly, the plane in which [the form of point] A is reflected to [point] T constitutes a cylindric section [because it is oblique].

[2.517] Now, let TA be extended beyond T and A on both sides to form RP. Since there are four planes of reflection, because [reflection occurs] from four points, and since the two points T and A lie in each of them, RP will be common to the four planes of reflection. Moreover, each of these planes cuts the plane tangent to the mirror at a point on its own common section [with the mirror], but not on the same common section [as any other plane of reflection].¹⁴⁶ Line RP is normal to one of the four common sections, but not to two, for if it were perpendicular to the tangent plane [common to two or more common sections], it would thus reach the axis. Hence, the normals [dropped] from point T to these four common sections are distinct [from one another], and there is only one that passes through A.¹⁴⁷

[2.518] Moreover, the normal [within any of the planes of reflection] will either be parallel to the line of reflection or will intersect it beyond or

inside the mirror. If it is parallel, the point of reflection will be the image location, as has been demonstrated [in proposition 32], and since there are four points of reflection, there will be four images. If it intersects, then, since there are four normals there will be four intersections and four images.

[2.519] Furthermore, given a visible point and a [center]point of sight, the point of reflection will be found [as follows]. For example, let A be the visible point. Construct a plane passing through A cutting the cylinder parallel to the base, and it will form a circle. Either [center of sight] B is in the plane of this circle, or it is not. If it is, we will find the point of reflection on that circle as has been shown for the spherical concave [mirror in propositions 38, 47, and the variation on proposition 38 provided in note 135]. If it is not [in the plane of the circle], draw a perpendicular to the plane of this circle from point B, repeat the previous proof, and the point of reflection will be found.¹⁴⁸ Moreover, when both eyes are looking, one image will actually form two, but they will abut or overlap, so they will appear single.

[Concave Conical Mirrors]

[2.520] In concave conical mirrors, the common section of the plane of reflection and the surface of the mirror will be a line of longitude along the mirror, or it will be a conic section. If it is a line of longitude, the image-locations will lie in [i.e., behind] the mirror[’s reflecting surface].¹⁴⁹ If it is a conic section, the image-locations will sometimes lie beyond the center of sight, sometimes at the center of sight [itself], sometimes between the center of sight and the mirror, and sometimes behind the mirror, just as was shown in the case of the concave cylindrical mirror.

[2.521] Furthermore, if a physical spot is taken on the normal extending from the center of sight to the plane tangent to the mirror [and if it lies] between the center of sight and the mirror, its form will not be reflected to the center of sight along the normal, because that spot will block the endpoint of the normal [at the eye], and for that reason it will not be reflected from it. However, if there is no such [physical] spot on that normal, [the form of] a visible point will be reflected to the eye along this normal, that point, and that point only, being the one on [the surface of the eye] that intersects the normal.

[2.522] On the other hand, if the center of sight lies on that normal as well as on the axis, it will form a circle, and the line drawn to any point on it from the center of sight will be normal to the plane tangent [to the mirror at that point], so from any point on that circle reflection can occur to the eye along the normal. And the portion of the eye that the two normals cut off to form the greatest [visible] angle on it will be reflected.

[2.523] If, however, the axis lies between the center of sight and the mirror, there will be no reflection to it along the normal except for that point on it that the normal intersects.¹⁵⁰

[2.524] **[PROPOSITION 53]** Now, if the center of sight and the visible point lie on the axis, [the form of] the latter will be reflected to the former.

[2.525] For instance, let H [figure 5.2.53, p. 286] be the center of sight, and T the visible point. Construct a plane that cuts the cone along the length of the axis, and let it be ABGH, with AH the axis and AB and AG edges of the cone. From point T draw TQ perpendicular to line AB, and extend it to QL. Let $[QL] = QT$. Then, from point H draw a line to point L, and it will intersect line of longitude AB. Let it intersect at point B, and from point B draw a line parallel to line TQ, which will necessarily reach the axis. Let it reach [the axis] at point D, and draw line TB.

[2.526] Since TQ is perpendicular to AB, and since $TQ = QL$, it is clear that triangle BTQ = triangle BQL, and angle QLB = angle QTB. But angle QTB = [alternate] angle TBD, and angle DBH = [alternate] angle QLB. Therefore, angle TBD = angle DBH, and so [the form of point] T is reflected to H from point B, and L is the image-location.

[2.527] Accordingly, if triangle TLH is rotated [about axis TH], point B will describe a circle on the cone, and from any point on that circle [the form of point] T will be reflected to [point] H. Meanwhile, outside this circle, L will describe a circle that will constitute in its entirety the image-location for point T.

[2.528] **[PROPOSITION 54]** Now, in this [sort of] mirror, having selected two points, i.e., Z and E [figure 5.2.54, p. 287] outside the normal [extending] from the center of sight and outside the axis, construct a plane on [point] Z parallel to the base [of the cone]. It will produce a circle in the mirror. E will lie in this circle or in another plane.

[CASE 1]

[2.529] Let it lie in the plane of that circle, and draw line EZ. It is evident [from proposition 49, case 3 above] that [the form of point] Z is reflected to E on one side of that circle from one point, or from two, or from three; and on the other side [it is reflected] from one [point].

[2.530] Take a point on the circle from which [the form of point Z] is reflected [to E], let it be H, and [let] T [be] the center of the circle. Draw lines ZH and EH. Diameter TH will bisect the angle [formed by them], and it will intersect line EZ. Let it intersect at point Q, let A be the vertex of the cone and AH a line of longitude.

[2.531] From point Q draw line QM falling orthogonally to line AH, and let it reach the axis [AT]. Let it fall at point D on the axis, and draw lines ZM and EM. From point Z in the plane of the circle draw line ZL parallel to line QH. Let EH intersect it. Let the intersection be at point L, and from point H draw HC perpendicular to LZ.

[2.532] Then, in the plane of triangle EMZ draw line ZO parallel to line QM. Let EM intersect it at point O, and draw line LO. From point C draw CN parallel to LO, and draw line NM.

[2.533] It is clear [by construction] that angle EHQ = angle QHZ as well as [alternate] angle HLZ [since ZL is parallel to QH by construction], and angle QHZ = [alternate] angle HZL. [Thus] HL = HZ, and HC is perpendicular to LZ [by construction. Since, therefore, HC is common to triangles LCH and CHZ] triangle LCH = triangle CHZ [by Euclid, I.26], and [so] LC = CZ.

[2.534] CN is parallel to OL [by construction, so, by Euclid, VI.2] LC:CZ = ON:NZ, so ON = NZ. Furthermore, since OZ is parallel to QM [by construction], plane ZLO will be parallel to plane QMH. Plane EOL intersects these two [planes] along [rectilinear] common sections, i.e., MH and OL, that will be parallel, so HM and CN are parallel [because CN is parallel to OL, by construction]. And because HC falls between parallels LZ and HQ, and since it is perpendicular to LZ, it will be perpendicular to HQ, so CH will be tangent to the circle.

[2.535] Therefore, plane AHC is a plane tangent to the cone. CN and NM lie in this plane, and line DM is perpendicular to this plane. It is therefore perpendicular to line NM, so NM is perpendicular to OZ, and [it has already been established that] ON = NZ. [Therefore, by Euclid, I.4] MO = MZ, and [so] EM:MO = EM:MZ [by Euclid, V.7].

[2.536] But EM:MO = EH:HL [by Euclid, VI.2], EH:HL = EH:HZ [by Euclid, V.7, because HL = HZ by previous conclusions], and EH:HZ = EQ:QZ [because HQ bisects angle EHZ]. Thus, EM:MZ = EQ:QZ [since EM:MZ = EM:MO = EH:HL = EH:HZ], so [because the respective sides of triangles EMQ and ZMQ are proportional, making the two triangles similar] angle EMQ = angle QMZ, [and] so [the form of point] Z is reflected to E from point M. Hence, if [the form of point] Z is reflected to E from point H on the circle, it is reflected to the same point from point M on the cone. And if [it is reflected] from two [points] on the circle, [it is reflected] from two [points] on the cone; if [it is reflected] from three [points on the circle, it is reflected] from three [points on the cone]; and if [it is reflected] from more [points on the circle, it is reflected] from more [points on the cone].¹⁵¹ On the other side of the circle, the proof that [reflection occurs] from one point on the cone just as from one [point] on the circle will be constructed in the same way.

[CASE 2]

[2.537] However, if E does not lie on the circle that passes through Z parallel to the base [of the cone], E will lie above or below it. Let it lie above, since the same proof applies to both cases. Draw line AE [figure 5.2.54a, p. 288] until it touches the plane of that circle, and let H be the point of contact, Q being the center of the circle. It is obvious that [the form of point] H can be reflected to Z from some point on the circle. Let it be T, and draw diameter QT [normal to the point of reflection]. Line HZ will intersect this diameter at point N. Draw [line] EZ and line of longitude AT.

[2.538] Since point Z lies on one side of diameter QT, and since [point] E lies on the other, it is evident that line EZ will intersect plane AQT. Let it intersect at point O, and from point O draw [line] OC perpendicular to line AT, and it will necessarily fall upon the axis. Let it fall at point D, and draw lines EC and ZC. I say that [the form of point] E is reflected to Z from point C.

[2.539] [Here is] the proof. From point Z draw line ZF parallel to [line] QT, and extend line HT until it intersects it. Let the intersection be at point F. Likewise, from point Z draw [line] ZK parallel to line OC, and extend line EC until it intersects it. Let the intersection be at point K.

[2.540] Since line ZF is parallel to [line] QT [by construction], while [line] ZK is parallel to [line] OC [by construction], it is clear that plane ZKF will be parallel to plane OCT, which lies within plane AQT [since O is where HK intersects plane AQT]. Plane HFK intersects these two planes along lines CT and KF. Hence, CT and KF are parallel.

[2.541] From point T draw [line] TP perpendicular to line ZF. Since it falls between two parallels [i.e., TQ and ZF], it is obvious that it will be parallel to line NZ, and so it will be tangent to the circle [at point T].¹⁵² Hence, plane ATP is tangent to the cone along line [of longitude] AT, and line OC is perpendicular to this plane. Accordingly, plane ATQ will be orthogonal to plane ATP, and plane ATP intersects the two planes ATQ and ZKF, which are parallel. Thus, the common sections, one being CT, the other PI, are parallel. But it has already been shown that CT is parallel to KF. Hence, PI is parallel to KF.

[2.542] But it is evident that [since ZT falls between parallels NT and ZF] angle NTZ = [alternate] angle TZF, whereas angle HTN = [alternate] angle TFZ, and TP is perpendicular [to ZF. Therefore, by Euclid, I.26] FP = PZ. But FP:PZ = KI:IZ, [so, by Euclid, VI.2] KI = IZ.

[2.543] Now, if line CI is drawn, then, since plane ATPI is perpendicular to plane ZKF [which is parallel to plane ATQ], CI will be perpendicular to ZK, and angle CKZ = angle KZC. But angle ECO = angle CKZ [by Euclid, I.29, because OC and KZ are parallel, and EC cuts them both], and angle

$OCZ = [\text{alternate}] \text{ angle } CZK$, so $\text{angle } ECO = \text{angle } OCZ$. Hence, [the form of point] E is reflected to [point] Z from point C, which is what was proposed.

[2.544] Moreover, if another point is taken on the circle from which [the form of point] H is reflected to [point] Z, it will be demonstrated that [the form of point] E is reflected to Z from some point other than C on the cone. And if [the form of point] H is reflected to [point] Z from three points on the circle, [the form of point] E will be reflected to [point] Z from three [points] on the cone; if [it is reflected] from four [points on the circle], it will be reflected] from four [points on the cone].

[2.545] Furthermore, the point of reflection from which [the form of point] E is reflected to [point] Z is easy to find when the point on the circle from which [the form of] point H is reflected to [point] Z is found, and it will be found in the preceding way.

[2.546] If, however, it were claimed that [the form of] point E could be reflected to [point] Z from more than four points on the cone, it would be possible by repeating the earlier proof to show that [the form of] point H is reflected to [point] Z from more than four points on the circle, and in the case where [the form of] point E will happen to be reflected to [point] Z from however many points on the circle, or from one only, [the form of] point E will happen to be reflected to [point] Z from that many points on the cone, or from one only, and vice-versa. But if [this] contrary claim is made, it can be disproven in the preceding way [i.e., according to proposition 49].

[2.547] Hence, it is clear that some of the points [that are reflected] have a single image, some [have] two, some three, and some four, but no [more] than four are possible. In addition, when the mirror is exposed to both eyes, the same image will have different locations, but, because of its imperceptibility, this difference [in location] does not cause an error [in visual perception].

NOTES TO BOOK FIVE

¹Here Alhacen establishes the agenda for the first topical segment of book 5: to verify the cathetus-rule of image-location on the basis of both empirical observation and natural philosophical principles. Specifically, what is to be shown is that the image of any point-object invariably lies at the intersection of the line of reflection and the normal dropped from the point-object to the reflecting surface. As already established in book 4, these lines and their constituent points lie in a single plane that is orthogonal to the reflecting surface at the point of reflection, and with that in mind Alhacen sets out in the next thirty-nine paragraphs to show that the cathetus-rule applies to any reflection, no matter the shape of the reflecting surface or the location of the viewpoint.

²Obviously, in the case of the cone, unlike that of the straight thin rod, the fact that the point-to-point connection is orthogonal to the mirror cannot be seen directly; it must be inferred through what Alhacen later refers to as “sense-induction” (see note 19, p. 490 below). Alhacen’s choice of a cone both here and in subsequent experiments is dictated by his analysis in book 4 of the radiation of light according to cones.

³The observations discussed to this point are illustrated in figure 5.1, p. 215.

⁴The image of the rod will be compressed in a spherical mirror, and the sharper the curvature of that mirror, the greater the compression. Accordingly, the apparent distance of the mark on the rod’s image from the mirror’s surface will be less than the distance from that surface of the mark on the actual rod, but both distances will be proportional to the lengths of the two rods.

⁵In other words, even though the narrowing of the actual rod may be fairly gentle, the compression of the rod’s image will make that narrowing appear more radical than it actually is. Such distortions, which fall under the category of visual illusions for Alhacen, are dealt with in book 6 of *De aspectibus*.

⁶What Alhacen seems to have in mind here is moving the cone on the mirror until some portion toward its base is blocked from view by the mirror itself—i.e., so that it falls behind the “horizon” of the mirror. In that case, the image of the portion that remains visible over the mirror’s horizon will be visible, and it will appear to lie directly in line with that portion.

⁷The illusion to which Alhacen alludes here involves both the compression of images and the bowing of those images under most circumstances in convex cylindrical mirrors. Accordingly, if the rod is placed aslant to the mirror’s surface, or if a cone is placed on its surface, the relationship between the object itself and its image will be skewed by curving in the mirror. Adding to the complexity of image-formation in convex cylindrical mirrors is the fact, already discussed in book 4, that the plane of reflection can form three different sections on the cylinder’s surface

when it cuts it: a straight line along the longitude, a circle parallel to the bases of the cylinder, and an ellipse.

⁸As constructed earlier, the panel is 6 digits high, and the cylindrical mirror inserted into it is 3 digits high. Moreover, that mirror was inserted so that its midpoint along the midline of longitude on the panel lay precisely 3 digits above the bottom of the panel. Thus, the top of the mirror lies 4.5 digits above the bottom of the panel. But by construction the distance from the top surface of the bronze plaque to the top surface of the ring is 5 digits plus half a grain of barley. Accordingly, if the panel were inserted into the hollow of the ring to rest at the level of the bronze plaque, the top of the mirror would lie half a digit plus half a grain of barley below the top surface of the ring. It is clear from the subsequent experiment, however, that the mirror must protrude above that surface, so presumably we are meant to fill the hollow with wax an adequate distance above the level of the bronze plaque to raise the panel with its inserted mirror high enough that the top of the mirror will stand above the top surface of the cylinder.

⁹Presumably, the kind of ruler Alhacen has in mind is the one represented in the left-hand diagram of figure 4.3.5, p. 194.

¹⁰In fact, Alhacen has already established this point empirically and descriptively in 4, 5.19-5.24, pp. 47-50 above.

¹¹The experiment just completed is illustrated in the top diagram of figure 5.2, p. 216. The mirror is represented by the segment of the cylinder in which DRE represents the midline of longitude. Imagine this segment inserted into its panel so that the mirror protrudes above the top surface of the ring, the inner edge of which is represented by arc FG. The circle passing through point R on the cylinder represents the projection of the plane of this surface through the cylinder, so this circle and FG are in the same plane. Along the midline of the top surface of the ring we pose a ruler so that it reaches midpoint R of the mirror and so that its sharpened edge touches the surface of the mirror at point R, where tangent XRY and midline of longitude DRE intersect. The surface of the panel is therefore tangent to the mirror along line of longitude DRE. Then, along the edgeline of the ruler, we place a needle with a small white object O at its endpoint. The needle, which is clearly meant to serve as a guideline for observation, is therefore posed perpendicular to the plane of the panel, so it lies directly in line with point R and centerpoint C of the projected circle on the cylinder. That point, of course, lies on axis HU of the cylinder. Under these conditions, if we establish a line of sight along the needle with one eye that is raised slightly above the needle, we will see that object O and the needle lie directly in line with their images in the mirror. Hence, the image of O lies on the normal along which the needle lies. In that case, moreover, the plane of reflection will be defined by line OR and midline of longitude DR on the mirror, so the plane of reflection will be orthogonal to the plane tangent to the mirror along that line of longitude—i.e., the plane of the panel—which includes tangent XY to point of reflection R. If we shift our point of view to B or A and establish a line of sight in the plane of the top surface of the ring, we will see that the images of O and the needle lie directly in line with their actual counterparts. In this case, the plane of reflection is defined not by OR and midline of longitude DR but by OR and

tangent XY. Nonetheless, since the needle is perpendicular to the plane of the panel, which is orthogonal to the plane of reflection, the image lies on the normal to its surface, and, no matter where we place the center of sight along the top edge of the ring, the image of OR will lie directly in line with OR itself. With this experiment, then, Alhacen has verified the cathetus-rule in two of the three cases according to which the plane of reflection cuts the cylinder: i.e., when that cut forms a line of longitude on its surface or when it forms a circle.

¹²In this experiment, Alhacen instructs us to place the panel atop a wax triangle, as represented in the bottom diagram of figure 5.2, p. 216, where the base of the panel is attached to triangle K. This triangle is thick enough to hold the panel firmly upright at the appropriate tilt when it is inserted into the hollow of the ring. As before, the mirror protrudes above the top surface of the ring, which is represented by FG, but in this case the plane projected through FG cuts an elliptical section on the mirror's surface because of its slant. The rest of the experiment is the same as illustrated in the top diagram. Thus, the needle with its object is placed along OR, which is normal to the mirror at point R and passes through point C on the cylinder's axis. If we then look with one eye along OR, we will see the images of the object and the needle directly in line with their actual counterparts. And if we shift the line of sight to B or A, the images and the objects will still be in perfect alignment, so the cathetus-rule is verified for the third case according to which the plane of reflection cuts the cylinder: i.e., when that cut forms an ellipse on its surface.

¹³In other words, the perpendicular is the shortest possible line between the object-point and the mirror, and this is the line through the point of reflection and the axis. The relevance of this point becomes clear later on, when Alhacen explains why the image should appear on the normal rather than on some other line, an explanation that appeals to both natural economy and symmetry.

¹⁴Thus, as Alhacen showed in book 4, all planes other than those that cut circles on the cone's surface will be planes of reflection.

¹⁵As will become clear shortly, Alhacen considers only those images that spread out on the mirror's surface not to be perceived according to reality, because they do not represent their object in any recognizable way. Later on, Alhacen will explain that there are five cases of reflection in concave mirrors that involve five different relationships between the normal dropped from the object-point and the line of reflection. One of them involves no intersection, both the normal and the line of reflection being parallel. Of the remaining four, one intersection occurs at the center of sight itself, and another occurs behind the center of sight, where the image cannot be seen.

¹⁶Obviously, the more brightly illuminated the cone, the brighter its image, especially in iron mirrors.

¹⁷In this case, the eye is displaced only slightly from the vertex of the cone; otherwise, the image will appear not behind but in front of the mirror.

¹⁸Alhacen's point about images that are perceived in front of the mirror is illustrated in figure 5.3, p. 217. Arc AB represents a section of a spherical concave mirror. The cone with its vertex at centerpoint C of the sphere is placed to the side so

that rod DF can be stood directly along line ECDF, endpoint D of that rod lying below centerpoint C of the circle, and center of sight E lying above it. Accordingly, the viewer is to direct his attention to point R, where line CR falls between D and E until he sees the image of endpoint D of the rod in front of the mirror. The form of point D will have reached E by reflection along RE, and it will lie directly in line with centerpoint C of the sphere and object-point D, so line ECDF will form a diameter of the circle and will therefore be orthogonal to the mirror's surface at F. Accordingly, the image of D will be seen at the intersection of line of reflection ER and normal FE. This, of course, is a gross simplification of a much more complex situation, but the empirical results are roughly consonant with the geometrical idealization.

¹⁹By "sense-induction" Alhacen clearly means a process of trial-and-error sighting rather than a process of precise instrumentation.

²⁰Figure 5.4, p. 217, illustrates the experiment to this point. The mirror, whose midline of longitude is DRE, forms a segment of the cylinder. FG is a segment of the inner edge of the ring, and the middle circle passing through R and centered on C is the projection of the plane of the top surface of the ring onto the cylinder containing the mirror. Thus, centerpoint C of that circle lies between R and segment FG of the ring. The needle is placed along line OR within the plane of the circle centered on C and perpendicular to midline DRE of the mirror at point R. OR is therefore perpendicular to tangent XY at point R. If the center of sight is stationed in the plane formed by OR and DRE, then reflection will occur anywhere along line of longitude DRE. Thus, if the eye is raised just a bit above object O at the end of the mirror, it will see that object, the needle, point R on the mirror, the needle's image, and O's image directly in line with one another along perpendicular OR and its extension behind the mirror. On the other hand, if the eye is stationed to the side, along AR, for instance, then the image of O will appear in front of the mirror but along line OR, so that O, the needle, O's image, and point R in the mirror will appear directly in line with one another. Thus, in both these cases, the image will appear at the intersection of the line of reflection, which is necessarily the line along which the image is seen, and normal OCR.

²¹In this case, if the center of sight is still fixed at A in figure 5.4, p. 217, and if another needle, like OR, with a small white object on its end is placed on the top surface of the ring so that its point lies directly in line with R and perpendicular to line of longitude DRE, then the image of the object will appear in front of the mirror and in line with the object, the needle, and point R on the mirror.

²²See esp. 2, 3.86-3.91 and 2, 3.135-3.146, in Smith, *Alhacen's Theory*, 454-456 and 475-479.

²³As Alhacen remarked earlier, we perceive images as if they were objects. Hence, our judgment of their size and distance depends on size-distance invariance, according to which we judge the distance of a given object according to its perceived size, which decreases as it gets farther away; see, e.g., 2, 3.137, in *ibid.*, 475. So it is essentially by a judgment of the amount of the visual field subtended by the image, correlated with a knowledge of the actual size of the object represented by the image, that we come to a conclusion about its distance. That is why, for instance,

when we see ourselves in a plane mirror, our image appears to lie as far behind the mirror as we actually lie in front of it. This distance is twice the distance between our face and the mirror, since the image reaches us along both lines of incidence and lines of reflection.

²⁴Alhacen is arguing from the equality of triangles OAR and IAR in the top, left-hand diagram of figure 5.5, p. 218, O being the object-point and I being the image-point. Thus, since OA and IA are equal, by supposition, since AR is common, and since angles OAR and IAR are right, it follows by Euclid, I.4 that OR and IR are equal.

²⁵Alhacen's argument here is a simple one based on symmetry. Let E in the top left-hand diagram of figure 5.5, p. 218, represent the center of sight, O the object, AR the mirror's surface, and R the point of reflection. We have already established that the image is seen along the extension of line of reflection ER. If the image is located at I, where normal OA intersects ER extended, then the two triangles OAR and IAR will be equal, so sides RI and OR on them will be equal. Thus, I will be perceived to be the same size and lie the same distance from E as the object would if it were viewed from a distance composed of OR and RE. If the image lay beyond the normal at I', it would appear smaller and farther away than it should, whereas if it lay in front of the normal at I', it would appear larger and nearer than it should.

²⁶In referring to the center of the eye here, Alhacen means not the centerpoint of the eyeball, which constitutes the center of sight, but rather the point on the cornea through which the visual axis falls.

²⁷Let the circle centered on E in the top, right-hand diagram of figure 5.5, p. 218, represent the eye, the circle centered on C the mirror, arc AB the portion of the eye visible in the mirror, and arc A'B' the image of the eye seen in the mirror. If the form of the eye's center, where line EC intersects arc AB, were to radiate to the mirror along a line other than normal EC and reflect back along it, then the image of E seen along that line and through that point in the mirror would be asymmetrical with respect to the image of the eye as a whole. On the other hand, if the image of A were to be seen at A'' outside of normal AC, then the resulting image A''B' would be asymmetrical with respect to the image of the centerpoint. Hence, it is only when the image of each point on the eye lies at the intersection of the line of reflection and the normal through that point that the image maintains its symmetry with respect to the eye itself. As Alhacen remarks, if the form reflects back along the normal, then there is no point of intersection between the two coincident lines, which means that there is no defined image-location. At the end of proposition 2, p. 400, however, he resolves this problem by claiming that the points surrounding the center of the eye do have a definite image-location, so we infer the image-location of the eye's centerpoint from the context of the image-locations of those surrounding points.

²⁸In providing this caveat Alhacen appears to acknowledge the tentative nature of his argument from symmetry, which is essentially an aesthetic argument. In so doing, he also seems to be acknowledging the limits of certainty in human reason, even when it is carried out punctiliously.

²⁹The claim that an image in a spherical convex mirror can appear beyond or outside the mirror seems at first glance to indicate that such images are actually

projected outside the mirror in the way real images are projected from concave mirrors. In fact, Alhacen means no such thing. What he does mean is that, in certain cases, the intersection-point of the line of reflection and the normal dropped to the mirror's center from the object-point falls beyond the great circle formed on the mirror by the plane of reflection on the invisible side of the mirror. Thus, even though it may be technically formed outside the mirror, such an image is still seen within the visible portion of the reflecting surface. As Alhacen points out in the very next paragraph, the same holds for convex cylindrical and conical mirrors. These points will be clarified in detail later on in various propositions. What Alhacen means in saying that the faculty of sight does not perceive the location of any image that coincides with the mirror's surface by deduction (*syllogistice*) is that in such a case it does not need to estimate its distance behind or in front of the surface according to the process discussed earlier in paragraphs 2.33-2.34.

³⁰In all these cases, the image-location, whether beyond, at, or inside the mirror, is a function of where the image lies with respect to the common section of the mirror's surface and the plane of reflection. In all the manuscripts collated, the word I have translated as "inside" is *citra* rather than *infra*. This use of the term is puzzling, because *citra* is generally applied by Alhacen to those images that appear to lie "in front" of concave mirrors.

³¹Adapted from Ptolemy's analysis in *Optics*, IV.71 (Smith, *Ptolemy's Theory*, 195), the lower diagram in figure 5.5, p. 218, illustrates these points. Alhacen's analysis later in proposition 32 below is somewhat more complex. Let E be the center of sight, C the center of the mirror, and R the point of reflection. When point O₁ is chosen on line of incidence O₅R, its form reflects to E along RE, and its image I₁, where normal CO₁ intersects line of reflection ER, lies behind the mirror. When point O₂ is chosen on the same line of incidence, its form reflects to E along the same line of reflection, but its normal O₂C is parallel to the line of reflection, so there is no intersection. The reflection of the form of point O₃ yields an image at I₃, which is behind the center of sight, whereas the reflection of the form of point O₄ yields an image at center of sight E itself. Finally, the reflection of the form of O₅ yields an image at I₅, which lies between the center of sight and the mirror. Of these images, only I₁ and I₅ appear clearly; the rest are nebulous, so they do not represent their objects "as they exist in reality," according to Alhacen's brief discussion in paragraph 2.26 above.

³²That angle BGD = angle AGH follows from the supposition that angle of incidence BGE and angle of reflection AGE are equal. Since angles EGD and EGH are both right, by construction, then the remaining angles BGD and AGH must be equal.

³³This is essentially the rationale provided in paragraphs 2.38 and 2.39 above.

³⁴In this case, of course, line AB forms the common section of two intersecting planes, one of which intersects the mirror's surface orthogonally, the other of which necessarily intersects that surface obliquely. Hence, no line drawn from AB to the mirror's surface in the plane that intersect that surface obliquely can be orthogonal to that surface.

³⁵In other words, given a single line of incidence to a given point on the mirror, there cannot be two different lines of reflection from that point.

³⁶As applied to plane mirrors, these four propositions encapsulate the four main points that Alhacen intends to address for the remaining six mirrors in order from convex spherical, through convex cylindrical, convex conical, concave spherical, and concave cylindrical, to concave conical. Those points are: 1) how to find the image-location given the center of sight, the point-object, and the point of reflection; 2) how to find the point of reflection given the center of sight and the point-object; 3) how many images of a given point-object can be formed for a given center of sight; and 4) how the resulting image or images are perceived in binocular vision.

³⁷In other words, line HG, which is supposed not to reach the circle's center, and NG, which does, must both be perpendicular to line PE, so they must coincide.

³⁸The purpose of establishing that AN (the distance between object-point A and the mirror's centerpoint N) is to AE (the distance between object-point A and endpoint of tangency E) as DN (the distance between image-point D and the mirror's centerpoint N) is to DE (the distance between image-point D and endpoint of tangency E) become clear later in proposition 16, where the proportionality is instrumental in proving that, for any given object-point and any given center of sight facing a convex spherical mirror, there can be only one point of reflection. Hence, the endpoint of tangency is of ancillary—and, indeed, very limited—significance in comparison to the four cardinal points (i.e., the center of sight, the object-point, the point of reflection, and the endpoint of the normal) listed in 4, 5.10, 4, 5.26, and 4, 5.39, pp. 329, 335, and 340 above.

³⁹In other words, as shown at the end of proposition 2, the image-location of the point on the anterior surface of the eye where normal GD intersects it will be determined by the image-locations of the points on the eye's surface surrounding it.

⁴⁰The reason for proving this point is that, if line of incidence OH and KD did intersect on the side of O and K when extended, then KD would be the normal dropped from their point of intersection, so its image would lie at point K on the mirror's surface, where the line of reflection GH meets KD. The image would thus not be inside the mirror, as proposed.

⁴¹As in the previous case, so in this one, if line of incidence PI were to meet SD on the side of P and S, the image of their point of intersection would be at S on the mirror's surface rather than inside the mirror.

⁴²In other words, when GHK rotates about axis GD, point H on the sphere's surface will describe a circle with its center on GD, and for any point of reflection on the portion of the mirror cut off by that circle, up to the circumference described by H, the image seen from center of sight G will lie inside the sphere that defines the mirror.

⁴³In this case, then, the segment of the mirror cut off by the rotation of GB will be defined by point B of tangency, so the relevant section of the mirror is contained by the circle described by B on the outside and the circle described by H on the inside. Whether the image lies inside, outside, or on the surface of the sphere that defines the mirror depends upon both the location of the point of reflection within that segment and the location of the object-point.

⁴⁴Thus, as pointed out in note 29 above, although the image may lie outside the circle on the mirror defined by the plane of reflection, that image will always be seen in the visible portion of the mirror, never outside it.

⁴⁵In other words, if ED is to serve as the normal for some object-point radiating to the mirror along line of incidence QF, then ED must intersect QF at that point.

⁴⁶As specified in the previous proposition, this limit is determined by the equality of segment AT on diameter DA and FT on line of reflection BFT in figure 5.2.11, p. 225. As was demonstrated in that proposition, no point between T and A can serve as an image-location because none of the lines of incidence falling to points between F and H will meet normal AD on the side of D.

⁴⁷As defined previously, O is the inner limit according to the equality of MO and BO. T is the outer limit insofar as proposition 9 established that all image-locations lying at the mirror's surface or outside it must lie on diameters below point G of tangency, and it is obvious that no reflection can occur from G, since lines GT and AG of incidence and reflection can form no angle, being part of one continuous straight line.

⁴⁸That is, in order to get to N, it would either have to cut the circle somewhere below N or loop through tangent AGT below N.

⁴⁹The reasoning behind this *reductio ad absurdum* seems to be as follows. We have established by construction that $RL < BL$. Given that supposition, the object is to project a line of reflection from A to normal LB through arc GD such that the segment of that line between the arc and the normal is equal to the segment of the normal between B and the point where the line of reflection intersects it. This former segment will either fall to the left of RL, or it will fall to the right. Suppose it falls to the right, and let it be HF in figure 5.2.13, p. 226. By supposition, then, $HF = BF$, so $HF > LB$. But $HF < MO$, which $= LB$. Therefore, HF is both longer and shorter than BL, which is impossible. Consequently, the sought-after segment must lie to the left of RL. That there is such a segment to the left of RL follows from the fact that, as the line of reflection from A sweeps from R to D along arc RD, and from L to B along LB, the segments on it lying between the arc and normal LB increase to at least the length of DB. As those segments lengthen, however, the segment they cut off from BL on the side of B shorten. So there must be a point at which the one segment's lengthening counterbalances the other's shortening until the two segments become equal. The main thrust of this proof is to preempt the possibility that this point might lie outside the circle, because if it did, then from that point inward toward B, the relevant segment of the line of reflection projected from A to normal BL through arc GD would be longer than the resulting segment of the normal. It would therefore follow that no point on FB could be an image location for center of sight A, according to how the limit of image-location is defined in proposition 10. But that contradicts propositions 9-11, where it is established that, if reflection occurs from any point on arc GD, there will be images inside the mirror.

⁵⁰What Alhacen means here is that the visual faculty does not perceive where the various image-locations lie within the invisible portion of the mirror in relation to the great circle produced in that portion of the sphere by the plane of reflection. All it perceives is that the images lie behind the visible portion of the mirror.

⁵¹Consequently, by Euclid, I.4, triangles BNG and ANG are equal, so angle BGN = angle AGN, and so their adjacent angles are equal, one being the angle of incidence, the other the angle of reflection.

⁵²This operation can be done as follows according to figure 5.2.19b, p. 230. Construct a circle whose diameter = QE. Take some random diameter XY, and to endpoint X apply tangent ZX = AG. Then, from endpoint Z of that tangent draw line ZECQ through centerpoint C of the circle. From Euclid, III.36, it follows that the rectangle contained by diameter EQ + segment ZE and segment ZE (i.e., ZQ,ZE) is equal to the square on the tangent, which is equal to the square on AG, since tangent ZX was constructed equal to AG.

⁵³That AH = EZ follows from the fact that DA = QZ and that DA,AH = AG², as just concluded. But we established earlier by construction that QZ,ZE = AG². Hence, DA,AH = QZ,ZE, and since DA = QZ, by construction, then AH = ZE. Moreover, since DA = DH + AH, since DA = QZ, and since AH = ZE, it follows that DA - AH = DH = QZ - ZE = QE.

⁵⁴For the relevant proposition in Apollonius' *Conics*, see R. Catesby Taliaferro, *On Conic Sections: Books I-III* (Encyclopedia Britannica, 1952), 685-686.

⁵⁵The relevant proposition is II.8, in *ibid.*, 687.

⁵⁶As previously established, $AD^2 = AD,DN + AD,AN$; but $GD^2 = AD,DN$; thus, $AD^2 = GD^2 + AD,AN$; also, $AD^2 = BD,DG = GD^2 + BG,GD$; thus, $AD^2 = GD^2 + BG,GD = GD^2 + AD,AN$; so $GD^2 + BG,GD = GD^2 + AD,AN$; subtracting GD^2 from both sides, we get $BG,DG = AD,AN$ or, reversed, $AD,AN = BG,DG$.

⁵⁷In fact, Euclid does not demonstrate this explicitly, but in III.36 he proves that, if any line, such as DA in figure 5.2.19d, p. 232, cuts a circle from a point outside it, the rectangle formed by the segments DH and HA produced by the cut will be equal to the square on the tangent dropped to the circle from that outside point. Since both DA and GB are projected from the same point D, their respective rectangles, DH,HA and DG,GB will be equal to the same tangent squared, so they will be equal.

⁵⁸We have established that $AD,DH = BD,DG$ and that $AD,DH = HD,DN + DH,AN$. Therefore, $BD,DG = HD,DN + DH,AN$. But we have also established that $HD,DN = GD^2$. Therefore, $BD,DG = GD^2 + DH,AN$, which is to say that $GD^2 = BD,DG - DH,AN$. But it has also been established that $BD,DG = BG,GD + GD^2$, which is to say that $GD^2 = BG,GD + GD^2 - DH,AN$, which finally translates to $GD^2 + DH,AN = GD^2 + BG,GD$. If we drop the common term GD^2 from both sides, we are thus left with $DH,AN = BG,GD$.

⁵⁹The point here is illustrated in figure 5.2.20, p. 233, by the two lines TC and TC' dropped from point T on section TP to section CU. In the first case, TC' = BG represents the minimal distance between point T on section TP and its matching branch CU. Hence, if the circle with radius TC' is produced from T, it will touch section CU at only one point. In the second case, the minimal distance between T and section CU is less than BG, so the circle with radius TC = BG produced from T will intersect section CU at two points, one such intersection being at point C on section CU. The two intersections at C and C' are represented in figure 5.2.20a, p. 234. The shortest possible distance between any two hyperbolic sections is the

apex-to-apex distance along the axis, so if T lies at the apex of section TP, and if the shortest distance between the two sections is equal to BG, then TC = BG must touch facing section CU at its apex, in which case TC will pass through the intersection of the two asymptotes. If, on the other hand, T lies at the apex of section TP but the apex-to-apex distance between TP and CU is less than BG, then TC = BG will intersect CU at two points. Finally, if T does not lie at the apex of TP, as represented in figure 5.2.20a, then the minimal distance between T and section CU must be no more than BG if TC = BG is to touch that section. If it is less, of course, then the circle with radius TC = BG produced from T will intersect section CU at two points.

⁶⁰As given in the actual manuscripts, figure 5.2.20 misrepresents this case, since all of them show CT intersecting section CU in such a way that the circle with radius TC = BG will cut CU at two points. Thus, contrary to what is specified here in the text, CT is not the minimal distance between T and section CU. There is no provision in the Latin text for insuring that the minimal distance between the hyperbolic sections produced not be greater than TC = BG, but according to Sabra, "Ibn al-Haytham's Lemmas," 310, the Arabic text refers the reader to V.34 and V.61 of Apollonius' *Conics*.

⁶¹We established earlier that $GB:BD = LM:MH$, so by reversal $BD:GB = MH:LM$. Hence, by Euclid, V.16, $BD:MH = GB:LM$. We just concluded that $BD:DE = MH:HZ$, so again, by Euclid, V.16, $BD:MH = DE:HZ$. Therefore, $GB:LM = DE:HZ$, and so, by Euclid, V.16, $GB:DE = LM:HZ$.

⁶²Alhacen's point here is illustrated in figure 5.2.20b, p. 234, which is an extension of figure 5.2.20. In addition to the original angles and the resulting line AED produced on the basis of line TC = BG passing above the intersection of asymptotes HL and NZ—as represented by the fainter lines—another line AED can be produced according to TC' = BG passing below the intersection of asymptotes HL and NZ—as represented by the darker lines. According to the resulting triangles, then, the previous proof will apply in this case as well.

⁶³GM will be a diameter since angle GDM is right, by construction, and is therefore subtended by a semicircular arc; so GM bisects the circle.

⁶⁴The sense of this passage in the Latin text is confusing, since it implies that the whole of line NL = H, but it becomes clear later in the proof of case one that it is not NL but its segment CL that is equal to H.

⁶⁵That these two triangles are similar follows from the fact that angle DMN + angle DCN = two right angles, as does angle TCL + angle DCN, so angle TCL = angle DMN, which = angle TAD, by construction. But vertical angles CTG and ATD are equal, so the remaining angles of each triangle—i.e., TLC and ADT—must be equal.

⁶⁶Although Alhacen's proof for case 2 is based on having D lie on BG beyond point B—i.e., beyond the right angle of triangle ABG—D can also lie on BG beyond point G, as illustrated in figure 5.2.21b, p. 235. In that case, the construction and proof are slightly different from those provided by Alhacen in cases 1 and 2. As before, we start by drawing DM parallel to AB so that angle GDM is right. We then extend AG to meet it at point M. Then, we form angle DMN equal to angle GAD and produce the circle through points M, N, and D. GM will thus be a diameter on

that circle, since it subtends right angle GDM. From point N we extend line NCL to meet line MGA at L such that $CL = H$ according to the required proportion $AD:H = E:Z$. Finally, we draw line CG and extend line DC to meet AB at point Q. DQ will intersect AG at T. It therefore remains to demonstrate that $TQ:TG = AD:LC$ (which $= H) = E:Z$.

The proof is as follows. Angle QAT = alternate angle GMD, since BA and MD are parallel by construction. Angle GCD + angle GMD = 2 right angles, by Euclid, III.22. But angle QAT + angle QAL = 2 right angles, by Euclid, I.13. Hence, angle QAL = angle GCD, so their adjacent angles GCT and QAT are equal. Since corresponding angles GCT and QAT in triangles GTC and QAT are equal, and since vertical angles GTC and QTA in those same triangles are equal, the remaining angles are equal. Thus, the triangles are similar, by Euclid, VI.4, so it follows that $TA:CT = QT:TG$. Meantime, angle NMD = angle TAD, by construction. But angle NMD = angle NCD, since they are both subtended by the same arc ND, and angle NCD = vertical angle TCL. Thus, angles TCL and TAD in triangles TCL and TAD are equal, and angle CTL is common, so the remaining angles TLC and TDA are equal. Triangle TCL is therefore similar to triangle TAD, from which it follows that $TA:CT = AD:LC$. However, as established earlier, $TA:CT = QT:TG$, so $AD:LC = QT:TG$. But, since $LC = H$ by construction, and since $AD:H = E:Z$ by construction, then $QT:TG = E:Z$.

⁶⁷This follows from the previously established conclusion that triangles AEL and ELQ are similar, since they are similar to similar triangles KMN and IKN. Thus, angle EAL = angle EQZ. But, by construction, angle EQZ = angle QAT (i.e., LAT), since EQ and AT are parallel, so angle EAL = angle LAT. Therefore, since angle GAU = angle GAE, by construction, and since angle EAL = angle LAT, angle UAT = $2GAE + 2EAL = 2GAL$, so angle GAL is half of angle UAT.

⁶⁸Although a definition of compound ratio is provided in book 6 (def. 5) in some of the manuscripts of Euclid's *Elements*, both Heiberg and Heath concluded with good reason that this definition was interpolated later and is thus not authentically Euclidean. The definition, as given in English by Heath, is as follows: "A ratio is said to be compounded of ratios when the sizes of the ratios multiplied together makes some (?ratio, or size)." In VI.23 Euclid does advert to the compounding of ratios on the basis of three magnitudes, K, L, and M, magnitude L being the mean proportional. Accordingly, Euclid claims that "the ratio of K to M is compounded of the ratios of K to L and that of L to M" (T. L. Heath, trans., *The Thirteen Books of Euclid's Elements*, vol. 2 [New York: Dover, 1956], p. 248), which would translate to $K:M = (K:L):(L:M)$. Within the context of the theorem, which involves the ratio of two areas, what Euclid seems to be getting at is that $K:M = \text{parallelogram } K,L:\text{parallelogram } L,M$, so that $K:M = K,L:L,M$, which is to say that the ratio of the respective sides is the same as the ratio of the areas formed by each of those sides with a given length L. Therefore, the full expression of the compound ratio $K:M = (K:L):(L:M)$ seems to be reducible to $K:M = K,L:L,M$.

⁶⁹That OU is bisected by FL follows from the fact that triangles QFT and OFU are similar, by construction. Hence, since FL bisects base QT of triangle QFT, it must also bisect base OU of triangle OFU.

⁷⁰It has already been established that $EC:GD = TQ:PO$ and that $GD:DI = PO:OU$. Hence, by Euclid, V.22, it follows *ex aequali* that $EC:DI = TQ:OU$. Another rationale for this conclusion is based on the already-established fact that $GD = PO$. But $EC = TQ$, by construction. Hence, $EC:TQ = GD:PO$, which is to say that $EC:GD = TQ:PO$. But we just established that $GD:DI = PO:OU$, so, by Euclid, V.22, $EC:DI = TQ:OU$. At bottom, of course, this means that $DI = OU$.

⁷¹We have already established that $EC:DI = TN:UH$ and that $DI:GD = UH:UP$. Thus, from Euclid, V.22, it follows that $EC:GD = TN:UP$.

⁷²We know from previous conclusions that $EC:GD = TN:UP$, and we have just concluded that $GE:EC = PT:NT$, so, by Euclid, V.22, $GE:GD = PT:UP$.

⁷³The second case just described is illustrated in figure 5.2.24a, p. 238, where LN is still equal to H , but CLN intersects diameter MG above the circle's centerpoint, rather than below it, as in figure 5.2.24. The two cases thus reflect the situation illustrated in figure 5.2.20b, p. 234, where two lines can be dropped from point A to diameter GB such that both the segments ED produced between the diameter and the circle below the diameter are equal to given line HZ . The proof for this second case remains the same as it is for the first: triangles QNL and DQA will be similar, as will triangles TQA and NQG , so it will follow that $TQ:QG = AD:NL = AD:H$, since $H = NL$, by construction. This lemma is actually a subcase of proposition 21, lemma 3, case 1. In this instance, however, instead of intersecting side AB of triangle ABG on the side of A (as illustrated in figure 5.2.21, p. 235), the requisite line from D intersects it on the side of B —i.e., on the opposite side of AB . Hence, point Q , where it intersects side AG , lies inside rather than outside the circle. These differences notwithstanding, the aim of the construction—to cut side AG of triangle ABG in such a way that $TQ:GQ = AD:NL (= H) = E:Z$ —is precisely the same as in proposition 21, lemma 3, case 1.

⁷⁴We established earlier that $BD:HT = BG:GH$, so it follows from the equality of DH and HT that $BD:DH = BG:GH$. We also established that $HD:DL = HG:X$. Therefore, given that $BD:DL = (BD:DH):(DH:DL)$, we can substitute the relevant ratios in the compounded expression to get $BD:DL = (BG:GH):(HG:X)$.

⁷⁵The gist of Alhacen's argument here is that, since there actually is a point of reflection D when center of sight A and object-point B are positioned as illustrated, then there will necessarily be a line SP that will fulfill all the appropriate conditions for the construction. This line, however, will not be $S'P'$, which forms $S'KC$ less than a right angle. If we do suppose that the relevant angle is less than a right angle (e.g., $S'KC$ in figure 5.2.25a, p. 240), then let A and B lie in the same locations with respect to the mirror as in figure 5.2.25. According to the construction given at the beginning of the proposition, we start by dividing line MK at F so that $BG:AG = FM:FK$, as in figure 5.2.25a. Next, we bisect MK at O , and draw line CO perpendicular to it. From point K we draw KC to form angle OCK equal to half of angle BGA . Through F we draw line $S'FP'$ so that $S'P':P'K = BG:GD$. We then connect $S'K$ to form triangle $S'P'K$. Finally, from angle BGA we cut off an angle BGD equal to angle $S'P'K$. Point D , where GD intersects the circle, will be the sought-after reflection-point. However, as is clear from the figure, point D lies below AG , so

there can be no reflection from B to A. In order, therefore, for D to be located on the mirror between GA and GB, angle CKS must be greater than a right angle.

⁷⁶In this case, then, we can produce line SP so that the angle SKC it forms with CK is greater than a right angle and so that the appropriate conditions are fulfilled for the selection of reflection point D between AG and BG on the mirror.

⁷⁷In other words, if two angles, CKS and CKS' could be formed at K, both of them greater than a right angle, then it would be possible to cut from BGA two angles BGD and BGD' equal to angles SPK and S'P'K, respectively, such that both D and D' would lie on the mirror between AG and BG, which is impossible. At bottom, the size of angle CKS depends upon the size of angle AGB and the relative lengths of AG and BG, since the proportions that dictate the configuration of triangle SKC depend on those variables. Accordingly, if A and B are positioned in such a way that the line connecting them is tangent to, or cuts, the circle—in which case, of course, there can be no reflection—then angle CKS will never be greater than a right angle.

⁷⁸As will become clear shortly, Alhacen's qualification here ("by sense-deduction") is crucial, because, according to mathematical analysis, most object-points will not share precisely the same image-location for both eyes in spherical convex mirrors. However, the disparity in location is generally small enough that the two images are perceived as one by the visual faculty.

⁷⁹Alhacen's point in this paragraph is illustrated in figure 5.2.25b, p. 240. In the left-hand diagram, the circle centered on G represents a single plane of reflection through the mirror. That plane contains the two centers of sight, A, and A', the visible point B, and centerpoint G. BG is the normal, and D and D' represent the points of reflection for the respective centers of sight A and A'. Accordingly, for both eyes the image will lie at I on the normal. In the right-hand diagram, there are two planes of reflection intersecting along normal BG, so the image for both eyes will still lie at point I on the normal.

⁸⁰Several points are at issue in these brief paragraphs. First, in spherical convex mirrors, the arrangement of the parts of the image is the same as that of the object, although of course the image is distorted by the mirror's curvature—a point left unspoken by Alhacen. Second, when both eyes see something in a spherical convex mirror, the constituent images of points on that object may not lie at precisely the same location for each eye, but when the point viewed is disposed the same way for both eyes, then its image will lie at precisely the same location for each eye. Orientation is the key. If a cross-section of the object faces the eye directly so that the visual axes and the common axis converge on a point on it, then the images in both eyes will lie at precisely, or very nearly precisely, the same spot on the normal dropped from the object point to the mirror's center. Likewise, for any cross-section of an object that coincides with or is at least in line with the common axis, the object-points on that line will appear at precisely the same point on the normal for both eyes. For any other cross-section, however, the image location for a given point-object will not lie at precisely the same point on the normal for both eyes. Hence, as illustrated in figure 5.2.25c, p. 241, if O is a point on a cross-section of the visible surface of some visible object, if E and E' are the centers of the right and left eye, respectively, and if the given cross-section lies off to the side, then the angle of

reflection will be different for E than it is for E', as already demonstrated in proposition 17 above. That being the case, points I and I', where the respective lines of reflection intersect normal OC, will be different for both eyes. In order, therefore, for the image-location to be precisely the same for both eyes, the angle of reflection must be precisely the same for each eye. Nevertheless, as Alhacen points out, the disparity in image-locations is generally so small that the visual faculty ignores it, fusing the images into one image, the result, at worst, being a slight blurring or indistinctness in the resulting perception. For Alhacen's discussion of this process of image-fusion in direct vision, see *De aspectibus*, 3, 2.1-2.22, in Smith, *Alhacen's Theory*, 562-573.

⁸¹As was pointed out in book 4, the phrase "cylindric section" translates the Latin phrase *sectio columpnaris* (or *columpnalis*). This, of course, is the figure produced in a cylinder by an oblique planar cut—i.e., one that intersects the axis of the cylinder at a slant. In proposition 20 of *On the Section of a Cylinder*, the fourth-century mathematician Serenus demonstrates that such a cut yields a true ellipse, not simply an ellipse-like oval. Altogether, then, there are three different ways in which a cylinder can be cut by a plane of reflection that also cuts the axis: through a line of longitude (so that both the line of longitude and the axis lie in the same plane), along a circle (so that the plane is perpendicular to the axis), and along an ellipse (so that the plane cuts the axis at a slant). Rather than translate *sectio columpnaris* as "ellipse," I have chosen the more literal "cylindric section" in order to distinguish it from what Alhacen calls the *sectio pyramidalis*, or "conic section" proper.

⁸²This, of course, is the same proportionality that obtains for spherical convex mirrors, as demonstrated in proposition 7 above.

⁸³As already remarked, when Alhacen refers to images "outside" the mirror, he means images that lie beyond the common section of the plane of reflection and the mirror on the invisible side of the mirror. The image is still seen behind and within the outer boundary of the visible portion of the reflecting surface.

⁸⁴Reflection within the plane of a line of longitude is therefore governed by the rules pertaining to reflection from plane mirrors.

⁸⁵The supporting logic of this abbreviated *reductio ad absurdum* is as follows. If we assume that there is a point L of reflection on the cylinder other than G, and if, as just demonstrated, this point L cannot lie on circle GH itself, then it must lie outside the plane of circle GH. If a normal LI is dropped from this point to point I on axis CE, then, when extended beyond the cylinder, it must intersect line A'B', because it must lie in the same plane of reflection as that line. LI will therefore intersect A'B' at some point K. But the normal from any point, such as K, on line A'B' will necessarily also meet the axis at point E and will therefore lie in the plane of GH. Two normals KE and KI will have thus been dropped from point K to axis CED, which is impossible, so reflection can occur from A' to B' at no point other than G on the cylinder.

⁸⁶Implicit in this method for finding the point of reflection on convex cylindrical mirrors is Alhacen's general method for finding the point of reflection on convex conical, as well as concave cylindrical and conical mirrors. Given object-point B and center of sight A in figure 5.2.28a, p. 243, we pass a plane through A to cut the

cylinder along circle ZEI. Then, from point B we drop line BH parallel to the edge of the cylinder—and thus orthogonal to the plane of circle ZEI—so as to meet that plane at point H. By proposition 25 above, we find point Z of reflection on circle ZEI according to which H will reflect to A at equal angles. We then produce a line of longitude from Z and pass a plane through line AB orthogonal to the cylinder's surface. Point G, where this plane cuts the line of longitude will be the sought-after point of reflection.

⁸⁷Angle CFG = angle CNG, and right angle GCF = right angle GCN, so angle CGF = angle CGN. But angle CGF = alternate angle GMQ, and angle CGN = alternate angle GNQ. Thus, angle GMQ = angle GNQ, and right angle NQG = right angle MQC, so angle NGQ = angle MGQ.

⁸⁸In other words, if we take the circle in the abstract, not as an actual section of the cone, then, assuming that the plane tangent to point of reflection G is perpendicular to the plane of the circle, the form of M will reflect from G to N, because the normal TGQ will be orthogonal to that tangent plane. The ulterior purpose of the theorem to this point is to establish that, if the form of point B is assumed to reflect to A from point G, then, when a line parallel to line of longitude EG is dropped from B to point M on the plane of circle PG, and when a line parallel to line of longitude EG is likewise dropped from A to point N on the same plane, the form of M *would* be reflected to N from point G on the abstracted circle, which is what is intended by the qualification "barring interference from the cone" (*non impediēte piramide*). The procedure just outlined, which involves dropping line AN parallel to line of longitude EG from the center of sight to the plane of circle PG, and dropping its counterpart BM to that plane from object-point B, is, of course, the extension of the method described in note 86 above for convex cylindrical mirrors.

⁸⁹In other words, if the two planes containing A and the two normals intersect on AL, then, since B supposedly lies within the same two planes, and since it lies in the same plane as FL, the two planes would have to intersect along both AL and FL, which is impossible.

⁹⁰Here, of course, Alhacen is treating G abstractly as a point of reflection on some surface to which the plane of reflection is normal.

⁹¹In other words, if we generate the equivalent of triangle SAL in figure 5.2.31, p. 246, and then connect the appropriate lines from M and N, we can create the equivalent of triangle CAL within the equivalent of triangle BAL and, on that basis, prove that angle MEU = angle UEN. The equality of these angles is of course obvious from the construction, since the plane GTERU that passes through the line of longitude GE and axis GT bisects the entire figure HANGM, so all the component triangles, GMN, MEN, and HEA are bisected at their respective vertices G and E.

⁹²Here, again, Alhacen is treating the plane of reflection, within which the circle lies, in the abstract, as though it were normal to a reflecting surface, even though it is not normal in the actual case of the cone.

⁹³If center of sight B lies at point D, then of course the normal to line of longitude GE will be dropped from P, in which case the point where that normal will intersect ZGE—i.e., the point of reflection—will fall below E. The farther beyond D point B lies, the farther below point E the point of reflection will fall on GE. On the

other hand, as B descends along BG toward G, the point of reflection ascends above E along GE. Whatever the case, if reflection is to occur from A to B, it must be possible to pass a plane through them such as to intersect one of the lines of longitude on the lower cone along the orthogonal. As Alhacen demonstrates later in book 5, there will be only one point of reflection in the concave segment of the circular section formed in the upper cone. The actual procedure for determining that point is given in proposition 38, pp. 458-459.

⁹⁴As in the previous case, so in this one, it is obvious that plane PZER passing through the line of longitude and the axis of the cone bisects angles AZD, AED, and A'EH. Clearly, then, no matter where B lies on DBG, the resulting angles BEQ and AEQ will be equal.

⁹⁵The actual construction and proof of this proposition is as follows. Let M in figure 5.2.31d, p. 248, lie in plane MGN, which passes through vertex G of the cone parallel to its base-circle. Let A lie above that plane. Draw AM, generate the complementary cone GZY, and pass a plane through A parallel to plane MGN so as to produce circle ZY. From point M erect a line perpendicular to plane MGN, and let it intersect plane AZY at point B'. Find point Z from which B' will reflect to A, and draw line of longitude ZGX along both cones. From M draw a line parallel to ZGX, and let it intersect plane AZY at point B. From A draw a line parallel to ZGX, and let it intersect plane MNG at point N. Draw AB and NM, and produce ZR and GS to bisect angles BZA and MGN respectively. The plane RZGS formed by these lines and ZG intersects line AM at point Q. The normal dropped from point Q to line of longitude ZGX will intersect it at point E, which is the sought-after point of reflection for M and A. As Alhacen remarks, the proof that follows from this construction is essentially the same as the earlier ones based on the equality of the respective angles of all the triangles formed by planes AZGN, BZGM, and RZGS—culminating, of course, with the equality of angles AEQ and MEQ.

⁹⁶At this point Alhacen's general method for finding the point of reflection in a convex conical mirror is clear. First, the relevant line of longitude must be found. This depends on finding the point of reflection on the circle projected from the center of sight either on the cone from which reflection occurs, or on the upper extension of this cone. Second, the point where the plane containing the line of longitude and the cone's axis intersects the line joining the center of sight and the visible point must be determined. From that point a normal is dropped to the line of longitude on the cone, and the point where that normal intersects the line of longitude will be the point of reflection.

⁹⁷This point is illustrated in figure 5.2.32a, p. 250. In this case, MQ represents the visible spot, exaggerated in size for the purposes of clarity. As just demonstrated, the form of MQ's central point T reflects to center of sight A along line AE, which is parallel to normal TD dropped from T, so there is no definite image-point for T. On the other hand, the form of point M is reflected to center of sight A along NA, which intersects normal MD at L. Meantime, the form of endpoint Q reaches the center of sight A along GA, which intersects normal QD at O, beyond the eye. And finally, the form of point T reflects to center of sight A from the opposite side of the mirror along CA, which intersects normal TD between A and the mirror's

surface. The same holds for every point between M and T and T and Q. Accordingly, the visual faculty averages out the various image-locations behind and in front of the mirror to locate the image on the mirror's surface in much the same way that it locates the image of the eye's centerpoint seen along the visual axis according to the context of surrounding points on the eye's surface.

⁹⁸In theory, of course, since the eye lies on the normal passing through each point on the rod, the image-location for all those points will be the center of sight. In practice, however, the rod has breadth, like the sensible spot just discussed. Accordingly, the resultant overall image of the rod will be located, part by part, according to the average of composite image-locations. On that basis, the rod will be seen as a confused blur on the mirror's surface, thus taking the curved shape of that surface.

⁹⁹In other words, if the center of sight lies at the center of the mirror, then reflection to that point can only occur along the normals dropped from that point to the mirror's surface, and such normals lie within the visual cone with its vertex at the center of sight.

¹⁰⁰Hence, as illustrated in figure 5.2.32b, p. 250, if A is the center of sight, then the form of no point, such as O, on radius AD will reflect to A, because no matter what theoretical point of reflection R is chosen, the normal DR to that point will lie outside the angle ORA and will therefore not bisect it. On the other hand, the form of point O' on radius O'D can reflect to A, because some point R' of reflection can be found such that angle O'R'A can be bisected by normal DR'. But its image-location will lie at A, so it will not be properly perceived. The form of point O'' on radius O''D can also reflect to A, because some point R'' of reflection can be found such that angle O''R''A can be bisected by normal DR'', and its image-location will lie beyond the center of sight at I, so it too will not be properly perceived by the eye at A.

¹⁰¹In other words, what is to be demonstrated is that the distance EB between object-point B and the mirror's centerpoint E is to the distance between image-point H and centerpoint E as the distance between object-point B and endpoint of tangency T is to the distance between image-point H and endpoint of tangency T. This, of course, is the very same proportionality that was derived for convex spherical mirrors in proposition 7 (see figure 5.2.7, p. 223)—i.e., $AN:DN = AE:DE$ —where A is the object-point, N the mirror's centerpoint, D the image-point, and E the endpoint of tangency. The significance of this proportionality as applied to concave spherical mirrors becomes clear later in proposition 36, pp. 452-454.

¹⁰²As in the earlier analysis of image-locations in spherical, cylindrical, and conical convex mirrors, so in this analysis of concave spherical mirrors, "outside the mirror" means outside the sphere defining the mirror, not simply outside the reflecting surface. The same will of course hold for concave cylindrical and conical mirrors. The claim that reflection can occur from arcs DT and GQ whether A or B lies on the mirror's surface or outside it, as long as the other point lies inside the mirror, reveals the true intent of this theorem: to prove not that reflection *will* occur from arcs TD and GQ but that it *may* do so. That it may not occur becomes clear if we assume that A in figure 5.2.34a, p. 253, lies closer in toward E and that B lies far

above the mirror along the extension of diameter TE. For in that case, no matter where the line of incidence from B strikes the mirror, the angle it forms with the normal from E will be greater than the angle formed by the line of reflection from that point to A. Thus, whether reflection will or will not actually occur from arcs TD or GQ depends on the relative position of A and B vis-à-vis their respective diameters.

¹⁰³Since $EQ = QD + ED$, and $QD = QH + HD$, then $EQ:QD = (QD + ED):(QH + HD)$. But $EQ:QD = QD:QH$. Therefore, by Euclid, V.19, $EQ:QD = \text{remainder } ED \text{ (from } QD + ED):\text{remainder } HD \text{ (from } QH + HD)$, so $EQ:QD = ED:HD$. Since $EQ = QH + HD + ED$, and $QD = QH + HD$, the lengths of both EQ and QD are ultimately contingent on the length of QH, which is determined as follows. For the sake of convenience, let $QH = a$, $HD = b$, and $ED = c$. Thus, $EQ = a + b + c$, and $QD = a + b$, from which it follows that $(a + b + c):(a + b) = c:b$. By Euclid, VI.16, rectangle $(a + b + c)b = \text{rectangle } (a + b)c$, which translates to $ab + b^2 + bc = ac + bc$. Dropping the common term bc , we are left with $ab + b^2 = ac$, which can be expressed in the form $b^2 = ac - ab$, which is to say that the square on HD is equal to the difference between the rectangle formed by QH and ED and the rectangle formed by QH and HD—i.e., $HD^2 = QH(ED - HD)$. Since HD and ED are given, their difference $ED - HD$ is given. Call that difference X. Therefore, $HD^2 = QH \cdot X$, so it follows by Euclid, VI.17, that HD is the mean proportional between QH and X. Consequently, $QH:HD = HD:X$. Given both HD and X, then, QH can be found by Euclid, VI.11, and once it is found, then QE will be formed by adding it to HE.

¹⁰⁴I have changed the text here to reflect my numbering of the propositions. In the Latin manuscripts, the propositional designation is confused, in great part because none of the manuscripts actually numbers the propositions. Thus, three of the manuscripts seem to refer to the “26th” proposition, and three seem to prefer “ZG” to “26.” One manuscript has “36” instead of either “26” or “ZG,” and this designation in fact refers more or less appropriately to the diagram numbered 36 in its text. The fact that all the manuscripts have something here, whether the number “26” or the letter designation “ZG” (which closely resembles “26” in medieval orthography) indicates that either the Arabic text or the original Latin translation derived from it had some sort of equivalent designation at this point.

¹⁰⁵The issue Alhacen addresses at the end of this analysis is whether a given arrangement of E and H will allow the form of H to reflect to E. His answer is that, as long as $EA:AH > ED:DH$, such reflection can occur. The actual demonstration of this point, which is unclear as presented in the text, is contingent on whether a tangent Q'G can be drawn from point G to intersect the point where diameter ZDAQ meets the circle produced at Q. If so, then $EA:AH > Q'E:Q'H$, which is to say that $EA:AH > ED:DH$, since $Q'E:Q'H = ED:HD$. When reversed to read $ED:HD = Q'E:Q'H$, this latter proportionality follows directly from proposition 33, where it is established that in reflection from concave spherical mirrors the distance between the object-point and the mirror's centerpoint is to the distance between the image-point and the mirror's centerpoint as the distance between the object-point and the endpoint of tangency is to the distance between the image-point and the endpoint of tangency. In this case—which reflects the situation illustrated in figure 5.2.33c,

p. 252—Q' is the endpoint of tangency, T the object-point, H the image-point (which coincides with the center of sight), and D the mirror's centerpoint.

How this stricture (i.e., that $EA:AH$ must be less than $ED:DH$ if reflection is to occur) actually applies is easily understood in the context of figure 5.2.36b, p. 256. In the upper diagram of that figure, E and H are situated so that point Q lies closer to the mirror than before, in this case coinciding with point A on it. Point G, where the two circles intersect, should be the point of reflection, and, according to the construction, the line of incidence EG will be tangent to the circle centered on Q. The tangent to point G on the mirror will intersect diameter ZA at point Q', which is thus the endpoint of tangency, and it is clear that $EA:AH > Q'E:Q'H$, which = $ED:HD$. Hence, the stipulation will have been met, so G is indeed a legitimate point of reflection. In the lower diagram, however, the limiting case is illustrated. Here, Q coincides with H, leaving tangent GE parallel to diameter AZ (i.e., angles HDG and DGE sum up to a right angle). Accordingly, point E can never fall on AZ, so there can be no reflection. Moreover, the stipulation that $EA:AH > ED:HD$ is no longer met, because EA and ED would be infinitely long, and there is no proportionality between finite and infinite magnitudes. In practical terms, what this means is that, as long as point Q falls to the left of point H, there will be some point E on diameter AZ to which the form of E will reflect. If, on the other hand, point Q falls at or to the right of H, there will be no such point E on diameter AZ.

¹⁰⁶The point here is illustrated in figure 5.2.37a, p. 257, where T and H both lie on tangent KEF. In this case a line of incidence can reach from T to any point on arc EG, such as R, on the concave side of the circle (assuming no blockage on arc EB), but the resulting line of reflection RH will lie on the convex side of the circle, so reflection cannot occur.

¹⁰⁷Like proposition 18 above, which deals with reflection from convex spherical mirrors, this proposition is based on the special case in which the visible point and the center of sight are equidistant from the mirror's center, which is to say that they lie on the extensions of two intersecting diameters of the circle. Accordingly, the point of reflection is found simply by bisecting angle HDT. If, on the other hand $HD \neq TD$, and if the resulting angle HDT is bisected by ED, which is continued to point Z, then clearly angle HZL $\neq$ angle TZL. The next two cases deal with special cases of this special case, when the two points lie on intersecting diameters inside the mirror at points on those diameters that are equidistant from the center. The more general case, in which HD and DT are unequal, is addressed in proposition 38, where Alhacen applies a method for determining the reflection-point that is equivalent to the one adduced earlier in proposition 25 for convex spherical mirrors.

¹⁰⁸In other words, for all equidistant point-pairs on HD and TD, including H and T, which are defined by the perpendiculars dropped from E to BD and GD, reflection can occur from E and Z only. As will become clear in the next proposition, this restriction is lifted for equidistant point-pairs except H and T lying on HG and TB.

¹⁰⁹At this point the significance of the circle produced through centerpoint D and the point-pairs chosen on BD and DG (e.g., circle HDTO) becomes clear. As

long as that circle intersects the circle of the mirror within the arc BG defined by the intersecting diameters, there will be four points of reflection within the mirror. If that circle is tangent to the mirror, or if it intersects the mirror at or beyond B and G, then there will only be two points of reflection. In the latter case this is obvious from the fact that the intersection-points will lie either on arcs AB and QG, from which no reflection can occur, according to proposition 34, or within arc QA, from which only one reflection (i.e., from Z) can occur, according to case 1 above. As will become clear later on, when Alhacen deals with points on the intersecting diameters that are not equidistant from the center of the mirror, the number of possible points of reflection is contingent on whether and how the circle passing through those points and the center of the mirror intersects arc BG.

¹¹⁰Although the text fails to make this specification, this theorem is intended to prove that, if the two points T and H are taken on diameters ADG and BDQ' in figure 5.2.38, p. 260, and if they lie at different distances from centerpoint D of the mirror, the form of point T will reflect to H from only one point on arc AQ'. As will become clear in fairly short order, the form of T can reflect to H from as many as three points on arc GB. Altogether, then, the form of point H can reflect to T from as many as four points within mirror ABGQ', one such reflection occurring from arc AQ'.

¹¹¹That $QM:LM = F'O':IO'$ is explained as follows. First, it has already been established that triangle LMC is similar to triangle O'ID, and triangle C'O'H is similar to triangle KMN. Therefore, corresponding sides IO' and LM of triangles O'ID and CLM are proportional, as are corresponding sides MN and C'O' of triangles KMN and C'O'H. Therefore, $LM:MN = IO':C'O'$. So, too, $LM:(MN + LN) = IO':(C'O' + C'I)$. By construction, however, $QN = LN$, and $F'C' = IC'$. Therefore, by Euclid, V.18, $LM:(MN + QN) = IO':(F'C' + C'O')$. Accordingly, since $LM + QN = QM$, while $F'C' + C'O' = F'O'$, then $LM:QM = IO':F'O'$, and, finally, inverting the proportion, $QM:LM = F'O':IO'$.

¹¹²It has already been established that $HI:IU = HD:DT$. However, by compounding, $HI:IU = (HI:IP):(PI:IU)$, which is to say that $HI:IU = (HI,IP):(IP,IU)$. Meantime, we know that $HI:IP = HD:DP$, so, substituting $(HD:DP)$ in the foregoing proportion, we get $HI:IU = (HD:DP):(PI:IU)$. But, by compounding, $HD:DT = (HD:DP):(DP:DT)$, and from earlier conclusions we know that $HI:UI$ (which = $[HD:DP]:[PI:IU]$) = $HD:DT$. Accordingly, $HD:DT = (HD:DP):(PI:IU)$, but also $HD:DT = (HD:DP):(DP:DT)$, and so $(HD:DP):(DP:DT) = (HD:DP):(PI:IU)$. Dropping the equal term $HD:DP$ from both sides, then, we end up with $DP:DT = PI:IU$.

¹¹³The train of logic here is as follows. First, it is clear from figure 5.2.38 that $UIH = O'IU + PIO' + PID + DIH$. But since $PID = DIH$, then $UIH = O'IU + PIO' + 2PID$. Now, by previous conclusions, we know that $O'IU = O'IH$, which = $PIO' + PIH$, which = $PIO' + 2PID$, so $UIH = 2O'IU = 2PIO' + 4PID$. Meantime $PIU = UIH - 2PID = 2PIO' + 4PID - 2PID = 2PIO' + 2PID$, and $UIH - 2PID = O'IU + PIO'$. But $O'IU + PIO'$ (which = PIU) = $2PIO' + 2PID$ (which = $2DIO'$). Thus, $PIU = 2O'ID$. Furthermore, given that $O'ID = CLM$ (i.e., FLM), by construction, and given that FLM [i.e., FLQ] is half of ADT (i.e., TDP), by construction, then $TDP = 2FLM = 2O'ID = PIU$.

¹¹⁴The method Alhacen lays out in this proposition for determining the point of reflection on the opposite arc from the object-point and center of sight is clearly analogous to the one provided in proposition 25 above for finding the point of reflection on a convex spherical mirror no matter where the center of sight and object-point lie outside its surface. As will become clear in short order, the basic construction for finding any point of reflection in a concave spherical mirror is essentially the same, although the proof is not always equivalent. In the following two sub-theorems Alhacen will prove that, as determined by the method just articulated, the reflection-point on the opposite arc is unique.

¹¹⁵If we take HDN as a diameter forming two right angles, angle LDH = 2 right angles – angle NDL, whereas angle MDH = two right angles – angle MDN. But angle LDN < angle FDL, so it is even smaller than angle MDL, of which FDL is a part. Hence, angle LDH > angle MDH.

¹¹⁶That is, if the new angle is cut from DHL, then the side of it below HDN would intersect ML between N and L, thus skewing the disproportionality between LH:MH and LN:NH even more.

¹¹⁷The reason Alhacen takes the tack he does in this theorem becomes clear at this point, for, by proving that angle DHL, and thus DHT, must be less than angle DHM, he precludes the possibility that reflection between point-pairs, such as T and L that lie unequal distances from the center of the mirror, can occur from point Z. On the one hand, if DHL and DHT could be equal to DHM, then, if point H were to coincide with Z, point T would necessarily coincide with M, so T and L would be equidistant from D. But that contravenes the initial condition of the proposition—i.e., that T and L lie different distances from D. Moreover, if LHD and DHT could be greater than MHD, then, if Z were taken as the point of reflection, point T would lie between M and B. Thus, T and L would lie unequal distances from D and yet would reflect to one another from Z, which is impossible, since such reflection requires that T and L be equidistant from D. Reflection from point Z is thus reserved exclusively for points on diameters BQ and AG that are equidistant from D.

¹¹⁸In other words, if a line of reflection is drawn from H in figure 5.2.41a, p. 262, to any point other than T or G on arc TG, it will form an acute angle with the normal dropped to that point. Accordingly, an equal angle can be formed at that point on the other side of the normal, and any point on the resulting line of incidence can serve as an object-point. Line HT cannot, of course, be a line of reflection since angle BTH of tangency is a right angle. Likewise, there can be no line of incidence from point G, because such a line would be tangent.

¹¹⁹Here, and in subsequent propositions, I have chosen to translate the expression *angulus reflexionis* as “reflected angle” rather than “angle of reflection,” because what Alhacen has in mind is the angle formed at the point of reflection by the line of incidence and the line of reflection rather than the angle formed by the line of reflection and the normal dropped from the point of reflection. Thus, as illustrated in figure 5.2.42, p. 262, the reflected angle (= *angulus reflexionis* in Latin) is OEA, which is twice the angle of reflection BEA understood in its proper sense.

¹²⁰This particular step of adding angle FBA to both ABK and FBT applies to figure 5.2.42 only. In figure 5.2.42a, the common angle ABF is added to either ABK or FBT to yield KBF = TBA.

¹²¹That $FBK + FEA = 2$ right angles follows from the fact that angles BFE and BKE in triangles FBE and KBE are right. Therefore, the remaining angles of each triangle—i.e., FEB and FBE, KEB and EBK—sum up to a right angle, so $FEB + KEB (= FEA) + FBE + EBK (= FBK) = 2$ right angles.

¹²²The ostensible intent of this proposition is fairly trivial: to prove that, if a center of sight is chosen on some diameter within a great circle on the mirror, and if a line of reflection is extended from that point to the facing arc, a corresponding line of incidence can be drawn from the selected point of reflection such that the form of any point on that line will reflect to the center of sight from the point selected on the mirror. Thus, no matter what diameter the center of sight lies upon, reflection can occur to it from any point on the facing arc from an infinite number of points on an infinite number of diameters. The ulterior—and more significant—point of the proposition, however, is to establish a relationship between the angle formed by the line of reflection and the line of incidence (i.e., the reflected angle) and the angle adjacent to the angle directly facing the reflected angle. As will become clear in short order, this relationship determines both whether and where reflection may occur in the arc facing the object-point and the center of sight. For instance, in figures 5.2.42, 42a, and 42b, p. 262, it is clear that when the object-point and the center of sight are equidistant from the mirror's center, the adjacent angle is equal to the reflected angle. Thus, in figure 5.2.42, when FB (the distance between object-point F and centerpoint B) is equal to AB (the distance between center of sight A and centerpoint B), angle FBG adjacent to angle ABF is equal to reflected angle AEF. Likewise, in figure 5.2.42a and 42b, where the distance TB between object-point T and centerpoint B is equal to the distance AB between center of sight A and centerpoint B, angle TBG adjacent to TBA is equal to reflected angle AET. From this it is evident that when the adjacent angle and the reflected angle are equal, the object-point and the center of sight must lie equal distances from the centerpoint on their respective diameters. It is also evident that, when those distances are not equal, the adjacent angle will either be greater than or less than the reflected angle.

¹²³This proposition bespeaks the rigor of Alhacen's approach to mathematical proof. From the previous proposition (i.e., 42) it is easy to infer that, when the reflected angle ATB and the adjacent angle AGD are equal, the center of sight and the object-point will be equidistant from the center of curvature (see esp. paragraph 2.403). But, as Alhacen recognizes, such an inference is inductive rather than deductive—hence the need to supply a proper proof.

¹²⁴The purpose of the proof to this point is to demonstrate, first, that if the reflected angle—e.g., BTA in figure 5.2.44, p. 264—is less than angle AGD, and if the two points A and B that are mutually reflected lie at different distances from centerpoint G such that $BG > AG$, then the circle BAT passing through points B, T, and A, will pass above centerpoint G. It follows, therefore, that arc BO on circle BAT is equal to arc AO, O being the point where normal TG intersects the circle. Accordingly, line KO, which intersects line BA at its midpoint, bisects circle BAT and therefore intersects BA orthogonally. Line GN, which bisects angle BGA, passes through point F on AB. Since point F lies to the right of KO, which is perpendicular

to AB, it forms an obtuse angle GFK with AB. Hence, GN and OK will intersect beyond line BA. The crucial point underlying this demonstration is that, for any reflected angle ATB less than angle AGD, the circle passing through the three cardinal points A, B, and T will invariably pass above centerpoint G of the mirror. From this it invariably follows that the relevant lines OK and GN will intersect above line AB. So let us assume that angle BQA is acute, as represented in figure 5.2.44a, p. 265, where it is in fact shown equal to angle ATB so that point Q lies on circle BTA, which passes above centerpoint G. Accordingly, by supposition, angle BQG = angle AQG, so point O', where normal QG intersects circle BAQT, will bisect arc AB, leaving arc AO' = arc BO'. According to the procedure described above, then, KO', which bisects AB, should intersect GF beyond line AB. But instead it intersects it below AB. Hence, point Q cannot be a point of reflection for A and B. Nor, for that matter, can any other point on arc EN, because, no matter what point is chosen on that arc to produce a reflected angle less than angle AGD, the circle passing through that point and points A and B will fall above centerpoint G, and the resulting point O', where the normal dropped from the chosen point intersects the resulting circle passing through it and points A and B, will lie to the right of G, the result being that O'K will intersect GN below AB. It bears noting that the intersection rule for reflected angle ATB less than angle AGD also holds for reflected angle AQB greater than AGB. In that case, however, the circle passing through A, Q, and B will fall below, rather than above centerpoint G. Thus, as illustrated in figure 5.2.44b, p. 266, when normal GQ is extended to intersect circle ABQ at point O', and when line O'K is drawn, it will coincide with line OK and will therefore intersect GN above AB at the very same point that OK does.

¹²⁵Under the conditions specified here for reflected angle ACB, the circle passing through A, B, and C must pass through point L, where KO and normal CG intersect if the resulting arcs AL and BL on circle ABT are to be equal. Circle ABC passing through C must therefore be larger than circle ABT, so it can only intersect circle ABT at the two points A and B. Yet, if it is to include C, circle ABC must somehow intersect circle ABT before reaching C from point B, and this is impossible. Furthermore, it is clear that, as C approaches N from T, point L, where normal CG intersects KO, moves upward toward K until it reaches a point where CG actually intersects KO above K. In all such cases, then—i.e., when C lies between T and N—the relevant circle through point L will be larger than ABT, so the proof will hold. On the other hand, if C is chosen between T and Z, CG will intersect the extension of KO below point O. The resulting circle that passes through A, B, and that point will thus be smaller than circle ABT, and so, in order to reach point C outside circle ABT, it must somehow pass outward through arc BT, which is impossible. Altogether, then, given points A and B lying at different distances from centerpoint G of the mirror, there is only one point T on arc ZE at which the angle of reflection ATB is less than angle AGD adjacent to angle AGB subtended by that arc.

¹²⁶Angle TML = angle TNL, because they are subtended by the same arc TL. But angle TNL > angle CTN, which = angle MTO, by construction. Hence, angle TML > angle MTO.

¹²⁷OTF = FTR, because MTF = NTF, since the respective arcs MD and ND that subtend them are equal, by construction. OTF = MTF + MTO, while FTR = NTF +

CTN. But $\text{MTO} = \text{CTN}$, by construction, so OTF and FTR are composed of correspondingly equal angles.

¹²⁸As will become clear in proposition 49, case 3, below, there are in fact three possible points of reflection on arc BG. However, only two of them can yield a reflected angle greater than angle ODA. Although not made explicit in the text, this is clearly the limitation intended in this proposition—i.e., that reflection between O and K can occur at only two points when the resulting reflected angle is greater than ODA.

¹²⁹From Euclid, III.22, we know that the opposite angles of any quadrilateral inscribed in a circle sum up to two right angles. Therefore, the point where ND intersects the smaller circle [i.e., s] must lie below D, because the complementary angle OsK formed by $s\text{O}$ and $s\text{K}$ must be smaller than ODK so the resulting angle $\text{OsK} + \text{OTK}$ will sum up to two right angles. That ND extended bisects arc OK follows from the equality of angles OTD and KTD , which are therefore subtended by equal arcs, Os and Ks .

¹³⁰If C were to fall between N and O, and if point x were to fall between N and C, i.e., to the right of N and to the left of C, it would necessarily intersect DC below D, because ND and CD intersect at D, and sx intersects ND below D. Thus, perpendicular sx would form a triangle with CD, one of whose base angles on the left is obtuse, since angle DCK is obtuse, by previous conclusions, and the other of whose base angles, sxC , is right.

¹³¹That is, ND and CD will form a triangle with its vertex at D and its two base angles DNO and DCK obtuse, since we have already determined that the angle at C on the side of K is obtuse. It follows, accordingly, that point x must lie between C and K, and C must lie between N and K. That being the case, then, angle KND must be acute, because line TD intersects line KO to the right of angle OCD, which is acute. Accordingly, TD will intersect sx , which is a radius of circle OTK, beyond point D at bisection-point s of arc OK. Were angle KND not acute, on the other hand, TD would never intersect sx . Furthermore, since sx intersects KO to the left of DC, and since angle DCK is obtuse, sx will intersect DU beyond line KO. Therefore, according to the conditions established in proposition 44, point T is in fact a legitimate point of reflection on arc BG for points K and O on diameters BD and GD.

¹³²This passage occurs at the end of paragraph 2.445 in all the manuscripts, but it clearly belongs here at the beginning of paragraph 4.451. The point of the passage is unclear, because it is unclear what is meant by an intersection below TO. If that intersection is taken to occur between D and O on line OD, then clearly TO and DO are unequal, since the one is a radius of the circle and the other is less than a radius. If the intersection is taken to occur below KD (and thus still below TO), then the passage is redundant.

¹³³KT:TF is compounded of KT:TL and TL:TF—i.e., $\text{KT:TF} = (\text{KT:TL}) : (\text{TL:TF})$ —but $\text{KT:TL} = \text{KD:DL}$, and $\text{TL:TF} = \text{DL:DO}$. KD:DO, in turn, is compounded of KD:DL and DL:DO—i.e., $(\text{KD:DL}) : (\text{DL:DO})$. Therefore, since $\text{KD:DL} = \text{KT:TL}$, and $\text{DL:DO} = \text{TL:TF}$, then $\text{KT:TF} = \text{KD:DO}$. The ulterior point here is to establish that, given KD and DT, for any point T that yields a reflected angle KTD greater than angle ODA, the given proportionality $\text{KT:TF} = \text{KD:DO}$ will hold. Alhacen's reason for establishing this proportionality becomes clear in the very next proposition, where it

forms the basis for verifying his method for finding a point of reflection on the arc facing points O and K when the reflected angle is greater than the adjacent angle.

¹³⁴In other words, KD and EH are given by construction, and TD is the radius of the circle, so all three are constant. As a result, MI is also constant, and it constitutes the fourth proportional, which can be found by Euclid, VI.12. According to MI's value, therefore, the endpoint F' of line F'M within circle TEC' is determinate.

¹³⁵The following explanation should make the force of this proof clear. Let point T in figure 5.2.46f, p. 272, represent one of two legitimate points of reflection on arc BG of the mirror such that the reflected angle—OTK in this case—is greater than angle ODA. The object is to find point F' and thus to define MI. To do this, we cut angle OTF = angle ODA from reflected angle OTK. We then bisect the remaining angle KTF with line TZ to find point E on diameter AG. From K we drop KH perpendicular to TZ so as to meet the extension of normal TD at point H. We then draw EC' parallel to KH, point C' being where that parallel intersects TH. Through points T, E, and C' we produce circle TEC' to intersect diameter AG at point M and then cut angle F'ME = angle EHD from angle TME. Line F'M will thus cut normal TD at point I to yield segment IM such that the proportionality KT:DT = EH:IM holds. We know from proposition 20, lemma 2, that two lines can be dropped from F' through diameter TD of circle TEC' to yield a segment equal to MI. This second line is F'N, which cuts normal TD at point K', so K'N = IM.

Now, let T' be another legitimate point of reflection on arc GB such that the reflected angle OT'K formed on it is also greater than ODA. This case is illustrated in figure 5.2.46g, p. 273, which is superimposed on the previous figure so that all but lines F'M and F'N of that previous figure are backgrounded in pale gray. Just as before, we cut angle OT'F from angle of reflection OT'K and bisect angle KT'F with T'Z' to locate point E', where line T'Z' intersects diameter AG. We drop KH' perpendicular to line T'Z' so that it intersects the extension of normal T'D at H'. Then we draw E'C'' parallel to KH' so as to intersect T'H' at point C''. Through the resulting points T', E', and C'' we produce circle T'E'C'', which intersects diameter AG at point M'. Taking angle F''M'E' = angle E'H'D, we cut it from angle T'M'E' to end up with line F''M', which intersects normal T'D at point I'. The resulting segment I'M' will therefore be such that KD:TD = E'H:I'M'. Since KD:TD = EH:IM, then E'H:I'M' = EH:IM. As in the previous case, so in this one, there is a second line F''P that intersects normal T''D at point L such that LP = I'M'.

If we turn, finally, to figure 5.2.46h, p. 274, which is abstracted from the previous one and somewhat enlarged, the implications of this analysis become clear. Take point T and its associated point F'. As we just established, F'N is the second of the two lines extending from point F' which, on passing through normal TD, produce segments equal to IM, K'N being the segment in this case. If from the second point of reflection T' we extend a line T'X parallel and equal to F'N, that line will intersect diameter ADG at point R such that T'R = K'N. That this is indeed the case can be easily understood if the production of line T'X is described in mechanical terms. Accordingly, as represented in figure 5.2.46k, p. 274, line F'N is shifted toward line T'X in the direction of the dotted arrows until point K' on F'N coincides with point T' and point N on F'N coincides with R. In this position, the line will

protrude above T' by the distance ZT' , which $= F'K'$. That line is then slid downward along $T'X$ until its upper endpoint Z coincides with T' and its lower endpoint N coincides with X . The corresponding distances ZT' and RX it will have moved in the process are equal to $F'K'$, so $RX = F'K'$. The remainder $T'R$ of the resulting line $T'X$ will thus be equal to $K'N$. As is evident from figure 5.2.46h, there is no point other than T' on arc BG at which a line extended equal and parallel to $F'N$ will intersect diameter AG in such a way that the segment between that point of intersection and the chosen point on arc BG will be equal to $K'N$, which $= MI$. Thus, from the perspective of T , T' is the only possible alternate point of reflection on arc GB that yields a reflected angle greater than ODA and fulfills the requisite proportionality $KD:DT = EH:IM$.

Now, take point T' and its associated point F'' in figure 5.2.46h. As established earlier, $F''P$ is the second of the two lines extending from point F'' which, on passing through normal $T'D$, produce segments equal to $I'M'$, LP being the segment in this case. If from the first point of reflection T we extend a line TY parallel and equal to $F''P$, that line will intersect circle TEC' at point S , the result being that $TS = LP$. Again, this can be easily understood in mechanical terms, line $F''P$ being shifted parallel to TY so that endpoints F'' and P coincide with T and Y , respectively. Thus, excess SY on TY that extends beyond circle $TEC' = F''L$, leaving $TS = LP$. As is clear from the diagram, if we choose any point on arc BG between B and x and drop the equivalent of line TY from it, TY will not intersect circle TEC' to yield a segment equal to LP . But at point x , which lies just above G , line xz extended parallel and equal to $F''P$ will cut circle TEC' at point y such that $xy = LP$. As Alhacen shows somewhat later, however, angle KxO formed at this point will be less than ODA . So, from the perspective of T' , T is the only possible alternate point of reflection on arc GB that yields a reflected angle greater than ODA and fulfills the requisite proportionality $KD:DT = EH:IM$. Altogether, then, T and T' are the only possible points of reflection on arc GB that yield reflected angles greater than ODA , which is what Alhacen set out to prove in the first place.

¹³⁶The requisite proof is based on cutting from angle OTK an angle OTF equal to angle ODA and extending TF until it meets KD at F . From there it is a matter of drawing on the previous theorem to demonstrate that $KT:TF = KD:DO$, which was shown to apply to any legitimate point of reflection on arc GB . Although Alhacen does not make it explicit, this same method will yield the second point of reflection. As was pointed out in proposition 25 above, a second line, $K''D''$ in figure 5.2.47a, p. 275, can be produced from line QH to line HT' such that $K''D'':D''T' = KD:DT$, the original $K'D'$ represented by the lighter line passing through E . Then, according to the procedure outlined in the proposition, we need only form angle KDT'' equal to angle $K''D''T'$ to find point T'' of reflection.

¹³⁷Given the construction in figure 5.2.49b, p. 277, it is possible for the form of B to reflect to A from part of the arc facing centerpoint G , and, since the reflected angle formed by them at any point on that arc is greater than adjacent angle BGD , it is possible for there to be as many as two possible points of reflection. Overall, then, in this case there can be as many as three reflections, two from the arc facing G and one from arc KD .

¹³⁸This point is illustrated in figure 5.2.49e, p. 278, which is based on the construction in proposition 47 above. Accordingly, O and K are the relevant points on diameters BD and GK, and $KD > OD$. As determined in proposition 47, T and T₁ are the two points of reflection on arc BG that form reflected angles OTK and OT₁K greater than angle ODA. If we pass a circle through points D, O, and K, it will cut arc BG at points E and F so that all the angles formed by points within arc EF will be greater than angle ODA, whereas all the angles formed within remaining arc-segments FB and EG will be less than angle ODA. Point T₂, which is the third legitimate point of reflection in arc BG, lies just below point E within arc EG, so it yields a reflected angle OT₂K less than angle ODA. That it cannot have a counterpart on arc FB follows from proposition 44 above, where it is demonstrated that within arc BG there cannot be two legitimate reflected angles smaller than angle ODA. Finally, in arc AQ' opposite BG, T₃ constitutes the only legitimate point of reflection. Altogether, then, O and K are situated within the mirror in such a way as to yield four, but no more than four, points of reflection.

To this point Alhacen has provided the method for locating points T and T₁ on arc BG (proposition 47) and point T₃ on arc AQ' (proposition 38), but he has not yet shown, nor will he show, how to locate point T₂ on arc BG. The method for determining this point is essentially the same as in proposition 38 above, albeit with a few adjustments. I will follow the format of that proposition as closely as possible for the sake of clarity and consistency.

Accordingly, let center of sight T on diameter BDQ' in figure 5.2.49f, p. 279, replace point O in figure 5.2.49e, let object-point H on diameter ADG replace K, let I replace T₂, and let $HD > TD$. Take some arbitrary line LQ and cut it at point K so that $QK:KL = TD:HD$, by Euclid, VI.10. Bisect LQ at point N, from point N draw perpendicular NF, and at point L form angle FLQ equal to half of angle BDG. Let FL intersect FN at point F, and from point K draw a line to side FL, intersecting it at point C such that $KC:CL = HD:DB$, by proposition 24, lemma 6 above. At centerpoint D of the mirror form angle IDB equal to angle LCK. That I is the sought-after point of reflection is demonstrated as follows.

At point I form angle O'ID equal to angle CLK. To the resulting line O'I drop perpendicular TC' from point T, extend line IC' to point F' so that $C'F' = C'I$, and draw lines TF' and TI.

Now, since angle O'DI (i.e., IDB) = angle LCK, by construction, and since angle O'ID = angle CLK, by construction, triangle CLK will be similar to triangle IO'D. Therefore angle IO'D = corresponding angle LKC. But angle C'O'T (i.e., IO'D) = angle LKC (i.e., NKM), and right angle TC'O' = right angle KNM, both being right by construction, so it follows that triangles NKM and C'TO' are similar, by Euclid, VI.4.

Now, if line ID is extended beyond D until it intersects the extension of C'T at point R, angle RDT = angle MCF, because angle MCF = two right angles – LCK, whereas angle RDT = two right angles – TDI, and $LCK = TDI$ by construction. Triangle RDT will thus be similar to triangle CMF, since angles CMF and RTD = vertical angles KMN and O'TC', respectively, which are corresponding angles in similar triangles NKM and C'TO'. Therefore, $RD:DT = FC:MC$, so, by Euclid, V.16, $DT:MC = RD:FC$.

But, since triangle $O'D$ is similar to triangle CKL , by previous conclusions, $KC:CL = O'D:DI$, so $KC:O'D = CL:DI$. Moreover, $KC = MC + MK$, and $O'D = DT + O'T$, so $(MC + MK):(DT + O'T) = CL:DI$, and so, by Euclid, V.17, $MC:DT = CL:DI$. But $MC:DT = FC:RD$ (given that $DT:MC = RD:FC$ by previous conclusions), so $FC:RD = CL:DI$, and so $RD:DI = FC:CL$. Accordingly, $RI:DI = FL:CL$, by Euclid, V.18. But $DI:IO' = CL:LK$, since triangle DIO' is similar to triangle CLK , by previous conclusions. Therefore, $RI:IO' = FL:LK$, by Euclid, V.22, so $IO':RI = LK:FL$. But $RI:IC' = FL:LN$, because triangle RIC' is similar to triangle FLN , since right angle $RC'I =$ right angle LNF , and angle $RIC' =$ corresponding angle FLN , by construction. Hence, by Euclid, V.22, $IO':IC' = LK:LN$, which reversed is $LN:LK = IC':IO'$. But, by construction, $IC' = F'C'$, and $QN = LN$, so $QN:LK = F'C':IO'$, and so, by Euclid, V.17, $QK:LK = F'O':IO'$.

Now, if line UI is drawn from point I parallel to TF' , and if line DB is extended until it intersects UI at point U , triangle $O'UI$ will be similar to triangle $TO'F'$, so angle $O'IU =$ angle $TF'O'$. Therefore, $TO':O'U = QK:KL$, because $TO':O'U = F'O':IO' = QK:LK$, by previous conclusions, and so $TO':O'U = TD:HD$, because $QK:KL = TD:HD$, by construction. But since triangle $TC'I =$ triangle $TC'F'$, because common side TC' is perpendicular to equal bases $F'C'$ and $C'I$, then angle $TF'C' =$ angle TIC' , so angle $TIC' =$ angle UIO' , which $=$ angle $TF'C'$, by previous conclusions. By Euclid, VI.3, therefore, $TO':O'U = TI:UI$, because angle UIT in triangle UIT is bisected by IO' , and so $TI:UI = TD:HD$, since both are as $TO':O'U$, by previous conclusions. At point I , finally, form angle $PID =$ angle DIT , and let P be a point on diameter DB .

It has just been established that $TI:UI = TD:HD$, and, by compounding, $TI:UI = (TI:IP):(IP:UI)$. By Euclid, VI.3, however, $TI:IP = TD:DP$, since angle TIP is bisected by DI , so, with $TD:DP$ substituted in the foregoing proportion, we get $TI:UI = (TD:DP):(IP:UI)$. Also, by compounding, $TD:HD = (TD:DP):(DP:HD)$, but we know from previous conclusions that $TI:UI$ (which $= [TD:DP]:[IP:UI]$) $= TD:HD$. Accordingly, $TD:HD = (TD:DP):(IP:UI)$, but also $TD:HD = (TD:DP):(DP:HD)$, and so $(TD:DP):(DP:HD) = (TD:DP):(IP:UI)$. With the equal term $TD:DP$ dropped from both sides, then, $DP:DH = IP:UI$.

Now angle $O'TT$ is half of angle UIT , by previous conclusions, whereas angle DIT is half of angle PIT , since angle $PID =$ angle DIT by construction. It follows that angle DIO' is half of angle PIU . But angle DIO' is half of angle HDP , since it is equal to angle FLQ , which, by construction, is equal to half of angle BDG , the vertical angle of HDP . Therefore, angle $PIU =$ angle HDP , and $DP:HD = PI:UI$, by previous conclusions. Hence, triangle UIP is similar to triangle HDP , by Euclid, VI.5, and so angle $UPI =$ angle HDP . Point H will thus lie on line PI , because angle $HPD +$ angle $HPQ' =$ two right angles, and angle $O'PI +$ angle $HPQ' =$ two right angles, HPQ' being adjacent to both HPD and $O'PI$. Therefore, since angle of reflection $TID =$ angle $PID =$ angle of incidence HID , the form of point H is reflected to point T from point I .

¹³⁹This last claim is based on proposition 37, cases 2 and 3, above. In case 2 it was demonstrated that, if the perpendiculars drawn from the two points H and T (in figure 5.2.37b, p. 258) converge at point E on facing arc BG , then E is the only

point of reflection within that arc. Likewise, H and T can reflect to one another from point Z, and only point Z, on the opposite arc QA. Altogether, then, only two reflections will occur. In case 3 it was demonstrated that, if EH and ET are not perpendicular to the diameters (in figure 5.2.37c, p. 259), reflection will occur from points M, E, and L, but from no others, on facing arc GB. Likewise, reflection will occur from point Z, and only point Z, on opposite arc QA. Altogether, then, four reflections will occur in this case. Since there is no intermediate case, it follows necessarily that, when the two points are equidistant from the center, it is impossible for there to be three, and only three, reflections.

¹⁴⁰In the cylinder at the top of figure 5.2.50a, p. 281, GA represents the normal common to both the circle GLA parallel to the bases and the ellipse FGK formed by an oblique cut along major axis FDK, GA being its minor axis. Accordingly, line X'GY' tangent to the ellipse at point G and line XGY tangent to the circle at the same point define the plane tangent to the cylinder along the line of longitude passing through G. It is impossible to represent this situation meaningfully from a perspective directly above the axis of the cylinder, because, in that case, the circle and the ellipse will coincide. Accordingly, I have represented the situation in figure 5.2.50 by inscribing the circle inside the ellipse, although in the manuscripts only the circle is given. Oddly enough, Risner represents the situation by having the circle circumscribe the ellipse so that the normal coincides with the major rather than the minor axis.

¹⁴¹Thus, as illustrated in figure 5.2.50a, p. 281, the form of point C will reflect from points H and L on circle ALGH, and, as shown in proposition 34, the form of any point on AD between A and D will reflect to E from two points on the circle on either side of AG. That there can only be two such points is demonstrated in proposition 36.

¹⁴²There is obviously some confusion in the text at this point. Point F will not be the image-location for a center of sight located at H and an object-point located at T, as specified in the previous paragraph. It will be the image-location for a center of sight lying at T if the object-point is at H. Although none of the manuscripts seems to recognize the problem, Risner makes the necessary adjustment, putting the center of sight at T where it belongs.

¹⁴³This point is easier to see if we abstract the case of reflection-point Q from figure 5.2.52, p. 283. Accordingly, in the segment of the cylinder provided in the top diagram of figure 5.2.52a, p. 284, points A and B, respectively, represent the object-point and the center of sight in the plane of the base circle, whose centerpoint is H. Within that plane E is a legitimate point of reflection, so normal EH bisects angle AEB, leaving angle of incidence AEH = angle of reflection BEH. If we drop a line orthogonal to the plane of the base circle at point B and choose some point T on it, that point will represent a corresponding center of sight, with Q the point on the mirror from which the form of point A will reflect to center of sight T. The circle of the plane within which Q lies is represented by the gray section passing through Q. Now, as this case is actually constructed, T lies above Q within a higher plane represented by the topmost gray section. Thus, the form of point A moves upward along AQ and, on reflecting from Q, moves yet upward along QT to the center of

sight. The figure to the right of the cylindric section represents a head-on view of the situation along a line of sight parallel to AH and just above the plane of the base-circle so that A lies directly behind H. Normal QU, which passes through AT at point I, bisects angle AQT so that angle of incidence AQU = angle of reflection TQU. In other words, quadrilateral AQTU with its diagonals TA and QU in the plane of reflection is simply a projection of quadrilateral AEBH with its diagonals BA and EH in the plane of the base-circle. Moreover, normal QU lies in the plane of the circle passing through Q, so it follows that the plane of reflection forms an elliptical section on the cylinder's surface with normal QU as its minor axis.

¹⁴⁴For example, the case of reflection-point L is represented in the lower set of diagrams in figure 5.2.52a, p. 284, both representing the same views as in the upper set. Point of reflection L lies in the plane of the circular section represented by the lower segment in gray, and center of sight T lies in the plane of the circular section above it. The normal in this case is LX, and the resulting quadrilateral ALTX with its diagonals LX and AT in the plane of reflection is a projection of quadrilateral ADBH with its diagonals DH and AB in the plane of the base-circle. Thus, angle of incidence ALX = angle of reflection TLX, normal LX lies in the plane of the circle passing through point L, and the plane of reflection forms an elliptical section on the cylinder's surface with normal LX as its minor axis. The same sort of analysis can be applied to M, the remaining point of reflection.

¹⁴⁵As before, the proof depends on extending a line from O to axis HU parallel to AC, which is parallel to HZ and lies in the same plane as OZ, which is parallel to HU, so that line will intersect the axis. Accordingly, the angle formed by that line and line of reflection OT will be equal to the angle formed by that line and line of incidence AO, so that line will be normal to point of reflection O on the mirror's surface within plane of reflection AOT.

¹⁴⁶That each point of reflection is unique to a given elliptical section on the cylinder's surface—and vice-versa—follows from proposition 50 above, where it is shown that reflection can occur from that point, and that point only, where the ellipse intersects the circle whose diameter forms its minor axis.

¹⁴⁷The various points in this paragraph are illustrated in figure 5.2.52b, p. 285. As we just established, each of the four points of reflection M, L, Q, and O lies in a unique plane of reflection that forms a particular elliptical section on the cylinder's surface. We also established that each of those points lies in a plane that forms a particular circular section on the cylinder's surface. These circles are stacked one above the other in ascending order, starting with the base circle and moving upward through the circles passing through points M, L, O, and Q. There are thus five circles in all. Furthermore, as we established, the diameter of each circle coincides with the minor axis of the ellipse particular to the given point of reflection. Finally, since A and T are common to all four planes of reflection, all four elliptical sections particular to their points of reflection must intersect along line AT, which is extended along RP.

Now, imagine the circle in figure 5.2.52b to represent the aforementioned stack of five circles viewed from directly above so that point M on its circle lies directly above point G on the base circle, point L directly above point D, point Q directly

above point E, and point O directly above point Z. Imagine, as well, that each elliptical section associated with its particular circle is rotated about its minor axis—which is the diameter of its associated circle—until it lies in the same plane as that circle. In the course of this rotation, the line coincident with RP on each of the elliptical sections will be moved until it comes to rest in a position slightly displaced from its original position coincident with RP, but for the sake of convenience we can ignore this displacement because it is so slight.

Accordingly, in figure 5.2.52b, each of the ellipses must be understood to lie in the plane of its associated circle in a stack of four. The lowermost ellipse *mn* is therefore associated with the circle passing through M, diameter MN of that circle being the minor axis of the ellipse, the next in line, *lk*, associated with the circle passing through L, its diameter LK being the minor axis, and so forth up the line from O (*op*) to Q (*qs*), the respective diameters / minor axes being OP and QS. Taking RP as common to all these ellipses, we can see that RP is normal to ellipse *qs* associated with point of reflection Q on the side of R and, furthermore, that it cannot be normal to any of the remaining three ellipses on the side of either R or P. Therefore, RP can be normal to only one of the four ellipses. Moreover, the only way it could be normal to more than one would be if it coincided with the minor or major axes of the relevant ellipses. That this cannot be the case is clear from the fact that RP does not pass through centerpoint H of the circle through which the minor and major axes of all the ellipses must pass. Or, as Alhacen observes, in order to be a normal common to more than one of the ellipses, RP would have to meet the axis, which passes through centerpoint H. Therefore, each ellipse has a distinct normal dropped to it from object-point A. The reason for establishing this point, of course, is to establish that for each ellipse—and thus for each plane of reflection—there is a distinct image, each image lying at the intersection of the particular normal and the particular line of reflection. Thus, in the case illustrated, the image of object-point A will lie at center of sight T, where normal RATP, dropped from object-point A to the reflecting surface, and line of reflection QT intersect.

¹⁴⁸In other words, if the center of sight were to lie at T, then we would reverse the process in the theorem to locate point B and then find the points of reflection in the base circle and then extrapolate from that. The method prescribed here is essentially the same as that prescribed for convex cylindrical mirrors.

¹⁴⁹That in the case of any conical mirror the common section of the plane of reflection and the mirror's surface cannot be a circle was made clear in book 4. If that section is a line of longitude, of course, then the rules governing both reflection and image-location will be precisely the same as those governing them in plane mirrors, so the image will always appear behind the reflecting surface.

¹⁵⁰Thus, as illustrated in figure 4.5.13, p. 212, if ARFB represents the concave conical mirror, and if the center of sight lies at vertex-point K of the lower cone, then all the lines of sight extended from K normal to the mirror's surface will strike it along circle RFB, so reflection will occur back to K along those same lines. Accordingly, all that is visible when the eye is so placed is that segment of the eye concentric with K from which the rays can reach the mirror and reflect back to K—i.e., the slice of the cornea bounded by the pupil through which the rays normal to

circle RFB can reach to the center of sight. On the other hand, if the center of sight is displaced from the axis, then only one line of sight will be normal to the mirror, so all that will be visible is the single point on the cornea through which that line of sight passes.

¹⁵¹Presumably, this latter qualification is meant to take into account the lone reflection from the arc on the opposite side of the axis, in which case, of course, the maximum number of possible reflections will be four.

¹⁵²The claim that TP is parallel to NZ is true if, and only if, points H and Z are equidistant from point Q, in which case QT will be perpendicular to HZ at point N and will bisect it at that point. As far as the proof is concerned, though, whether TP is parallel to NZ is irrelevant; all that matters is that TP be tangent to the circle. That it is in fact tangent follows from the fact that NT is parallel to ZF, by construction. Being perpendicular to ZF, then, TP is also perpendicular to normal QT and is thus tangent to the circle at point T.

**FIGURES FOR
INTRODUCTION
AND
LATIN TEXT**

FIGURES: INTRODUCTION

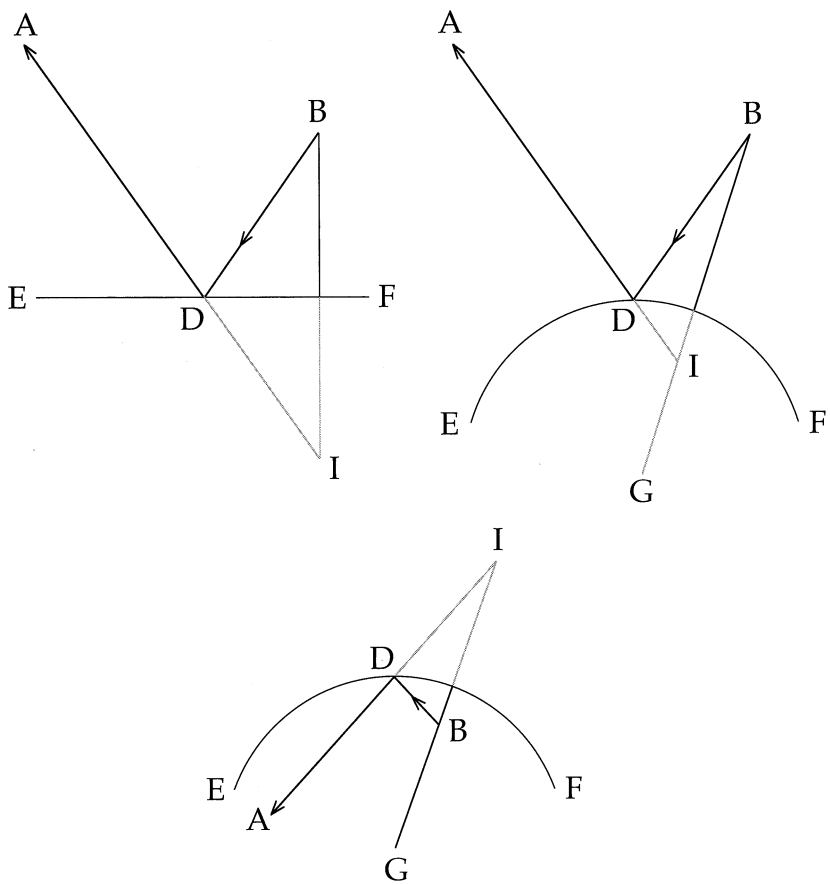


figure 1

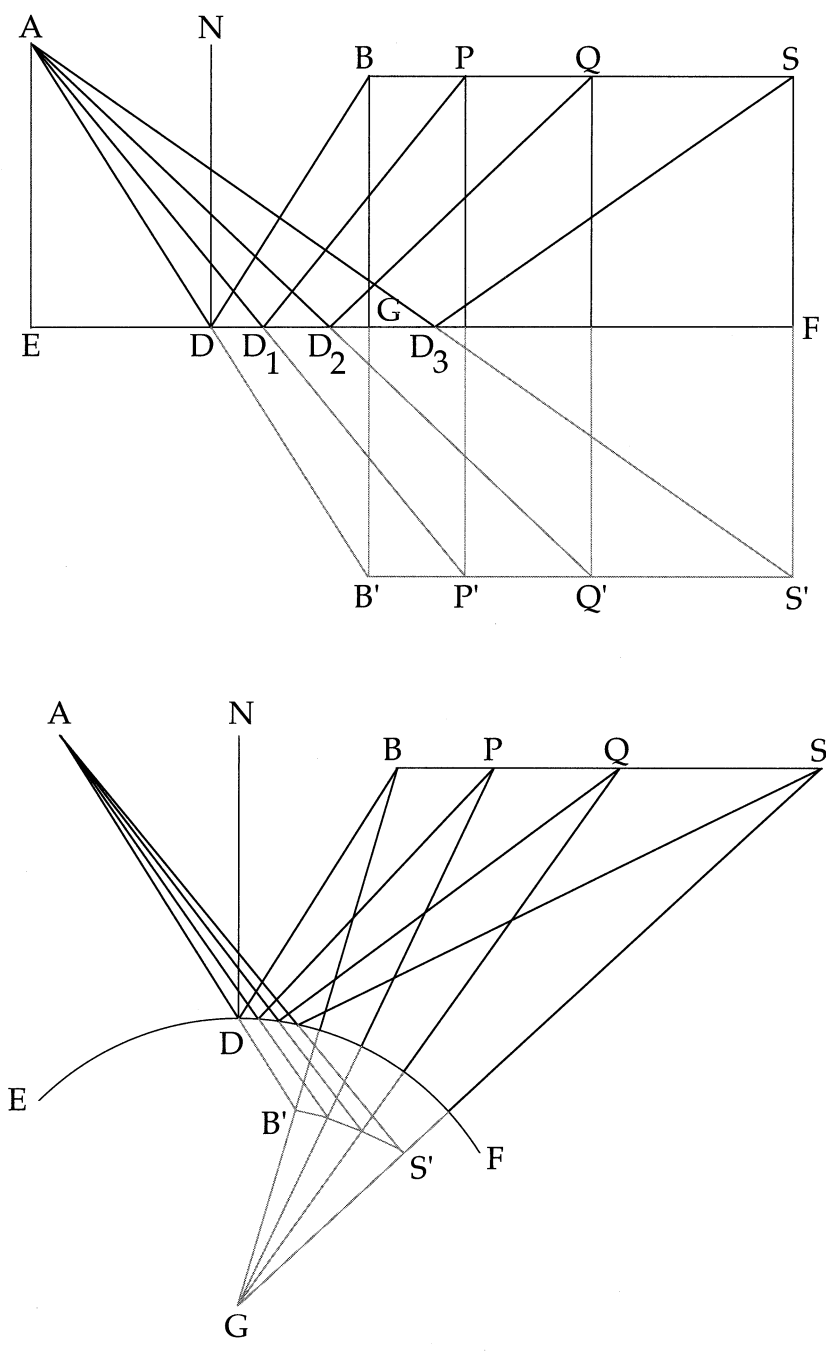


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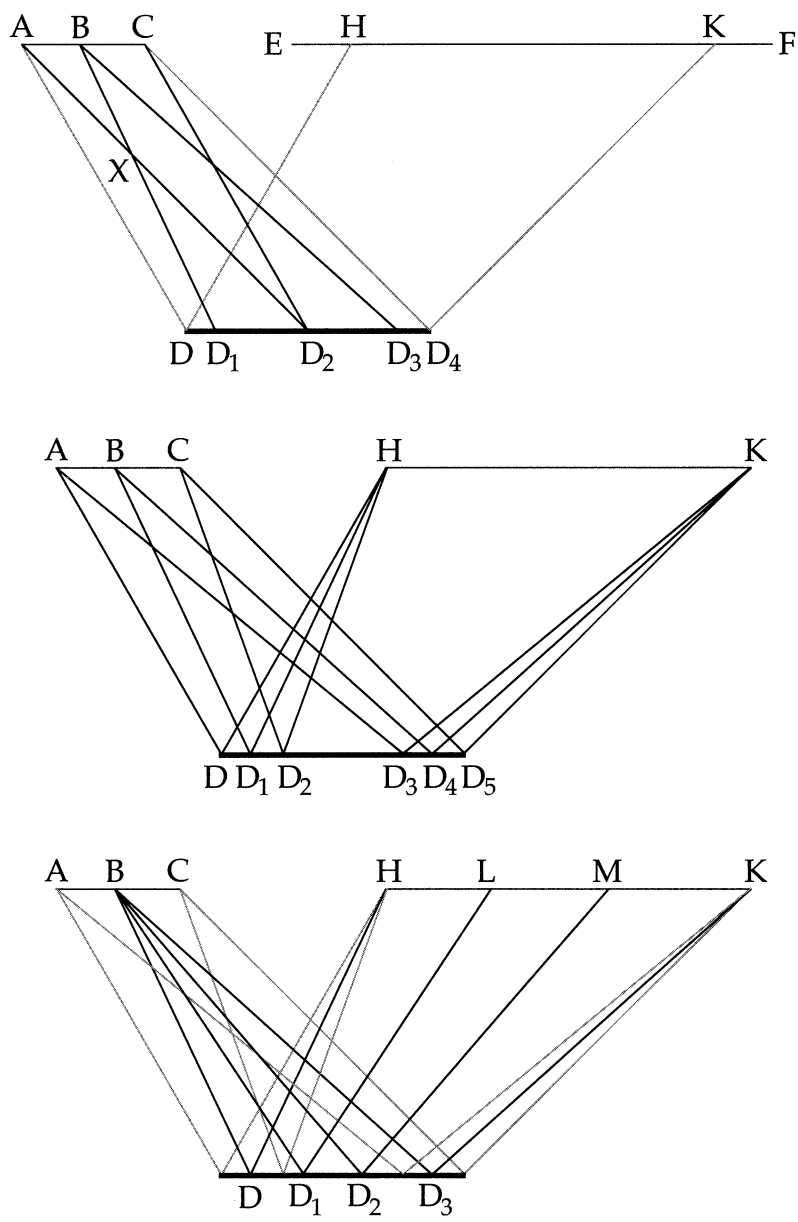


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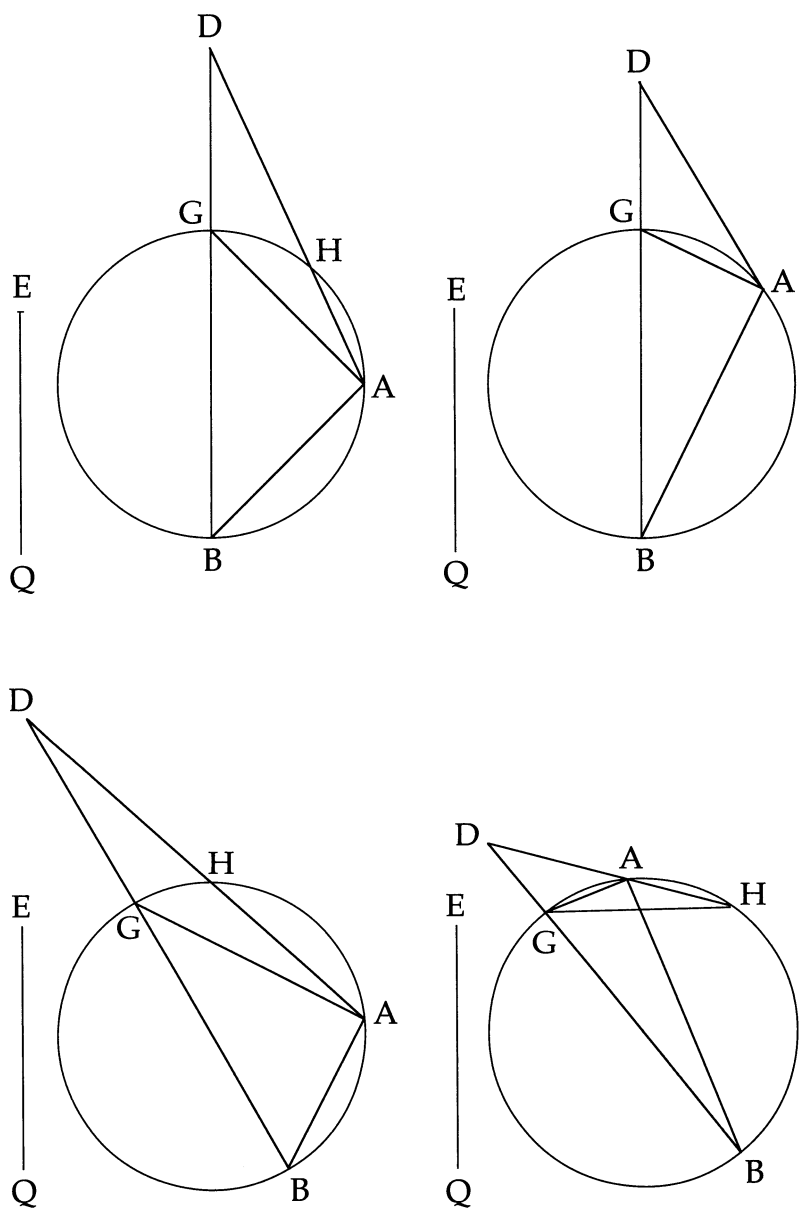


figure 4

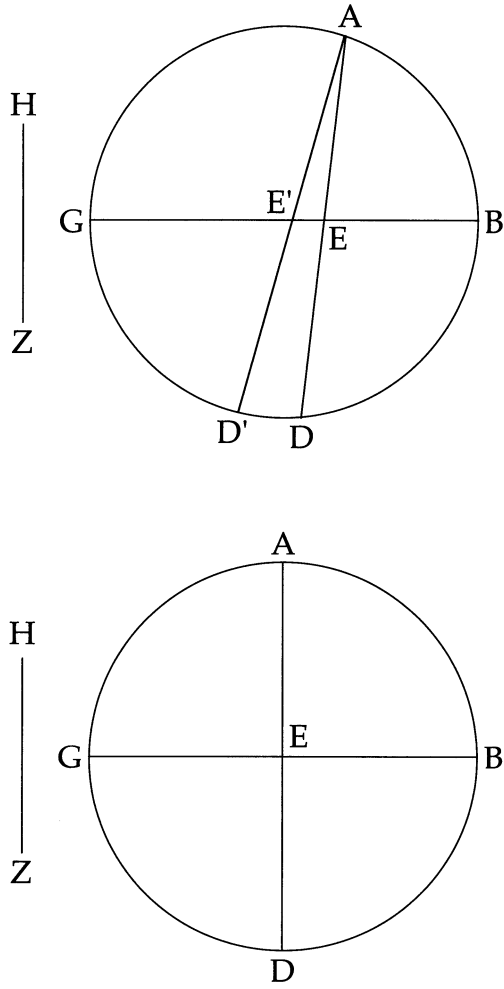


figure 5

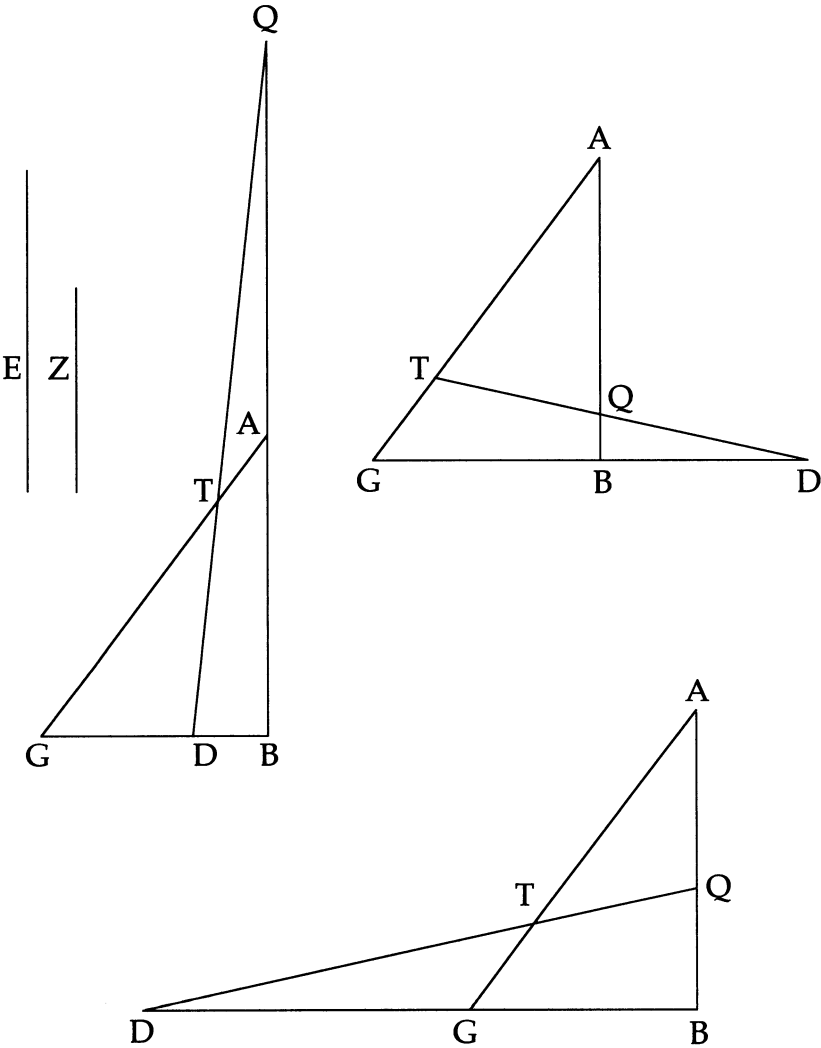


figure 6

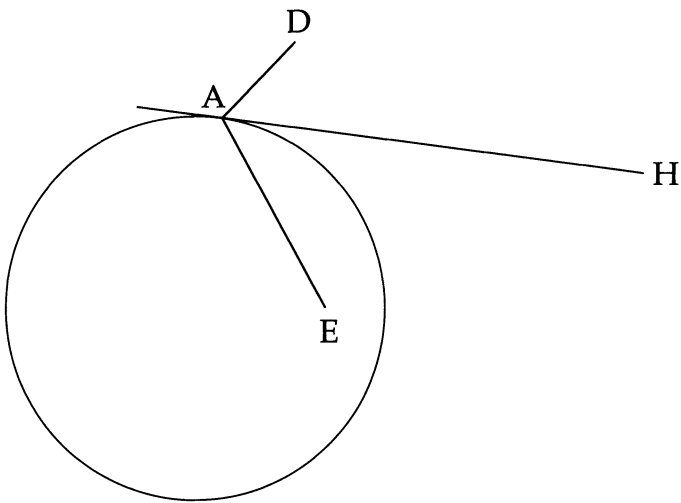


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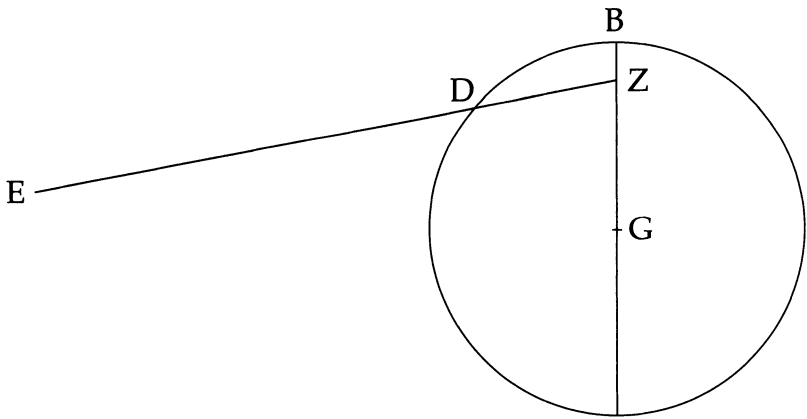


figure 8

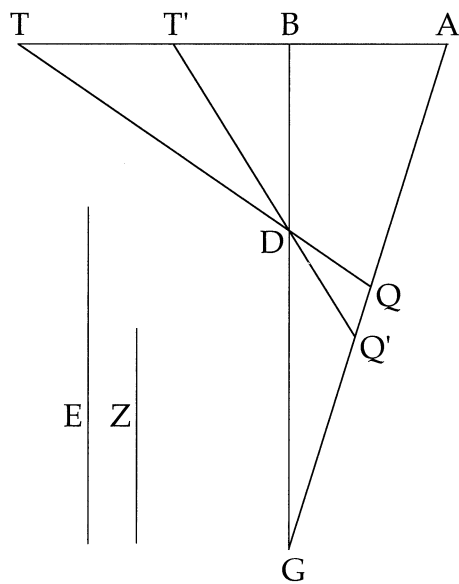


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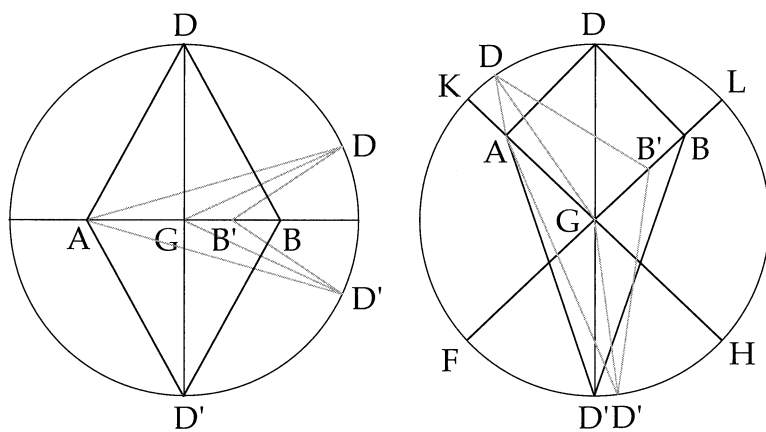


figure 10

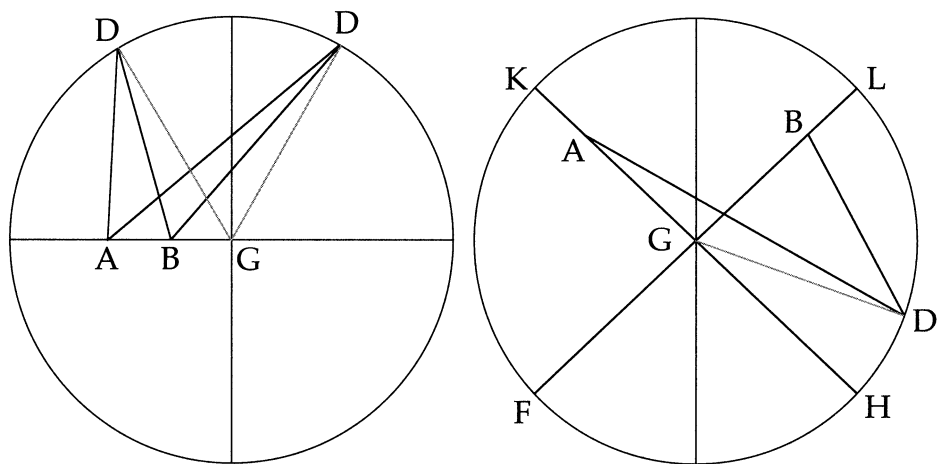


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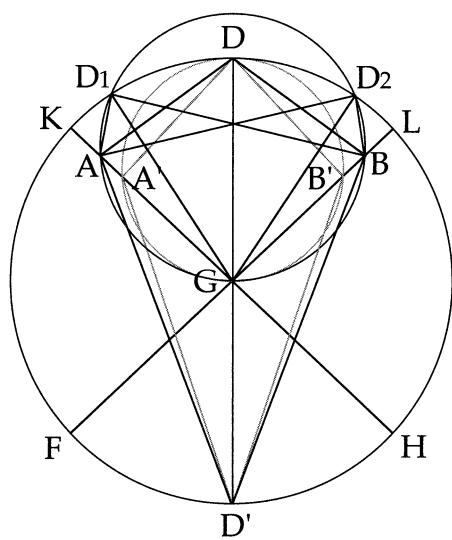


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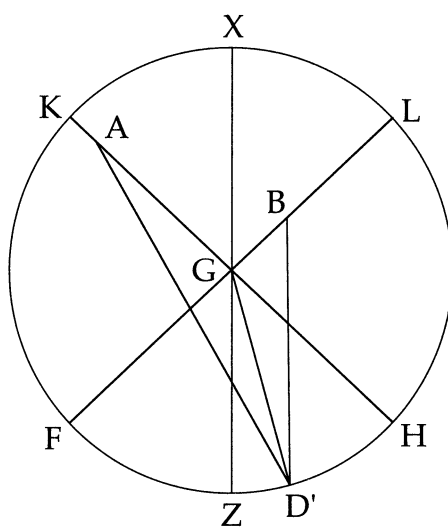


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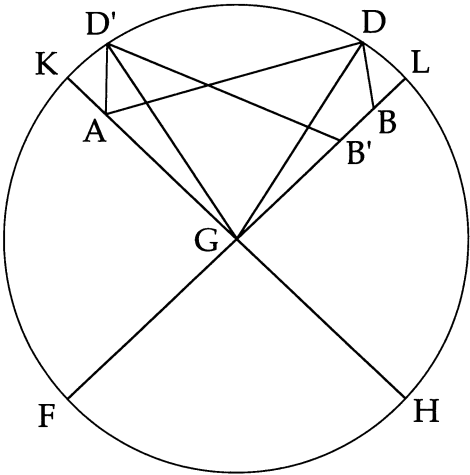


figure 14

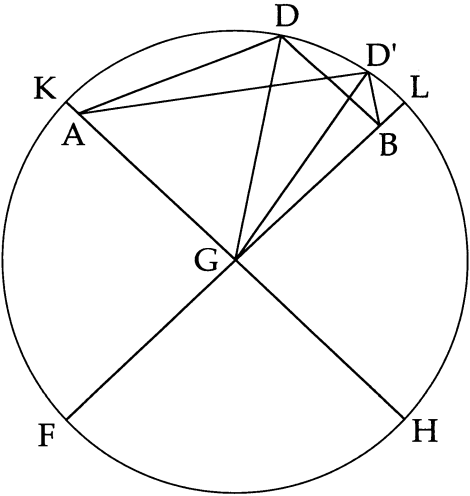


figure 15

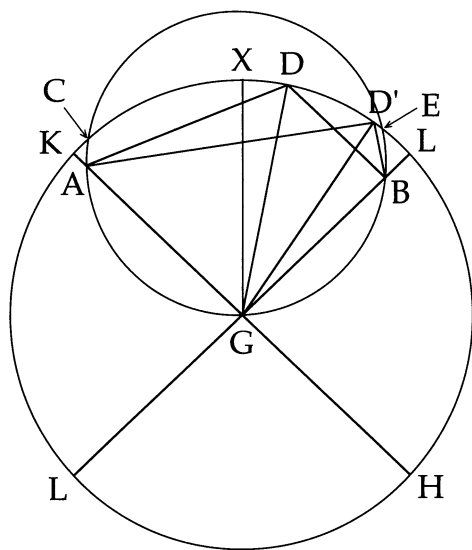


figure 16

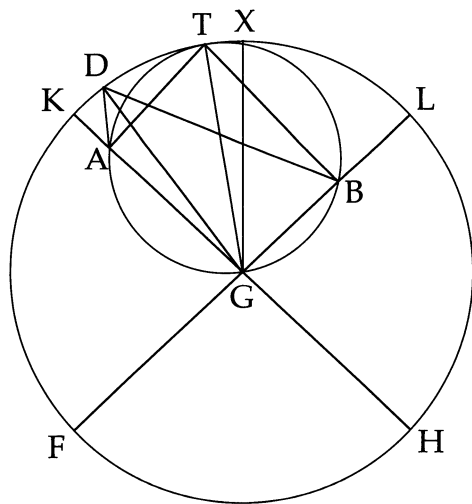


figure 16a

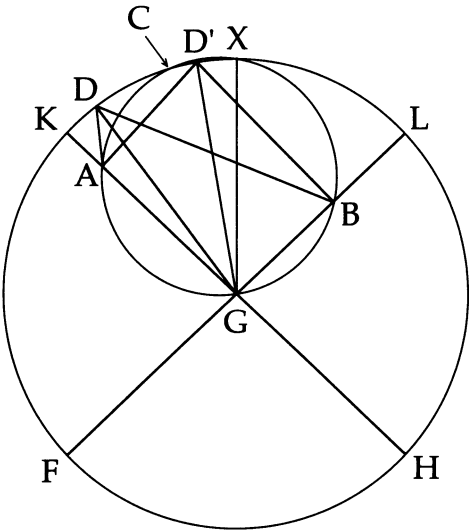


figure 16b

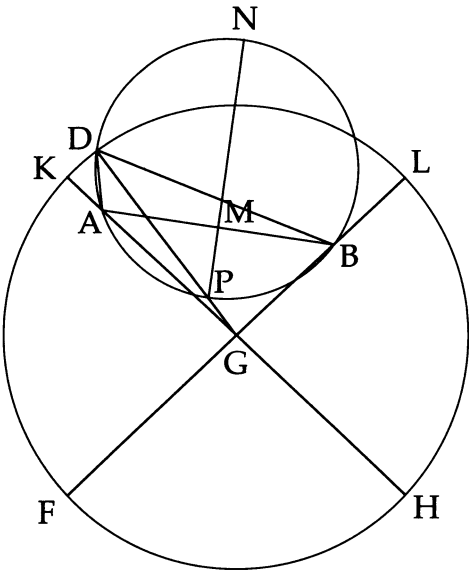


figure 16c

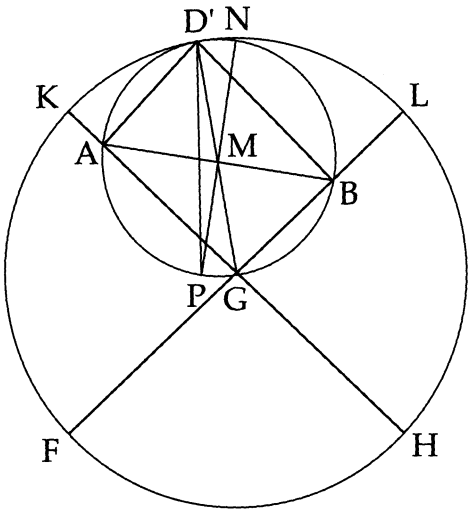


figure 16d

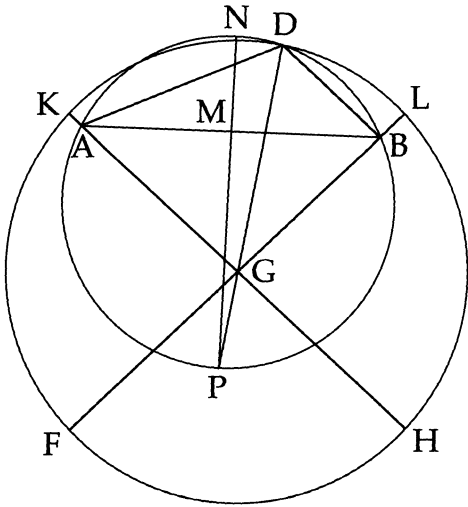


figure 16e

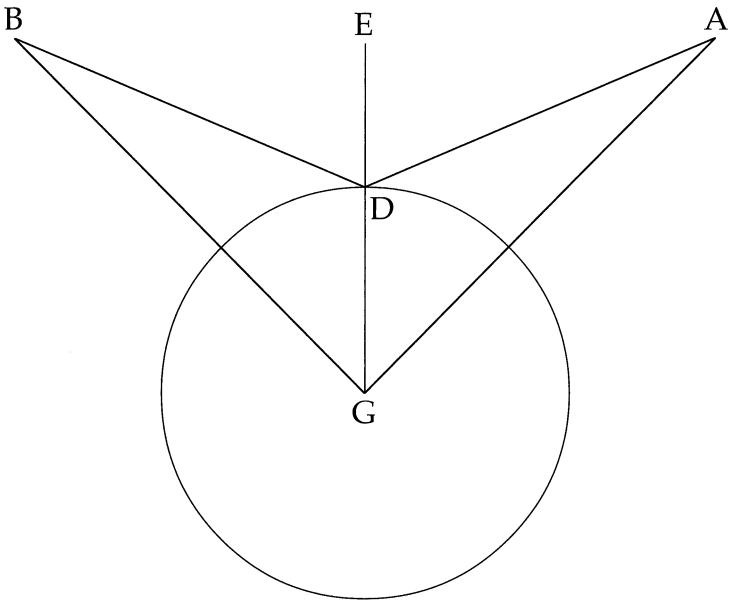


figure 17

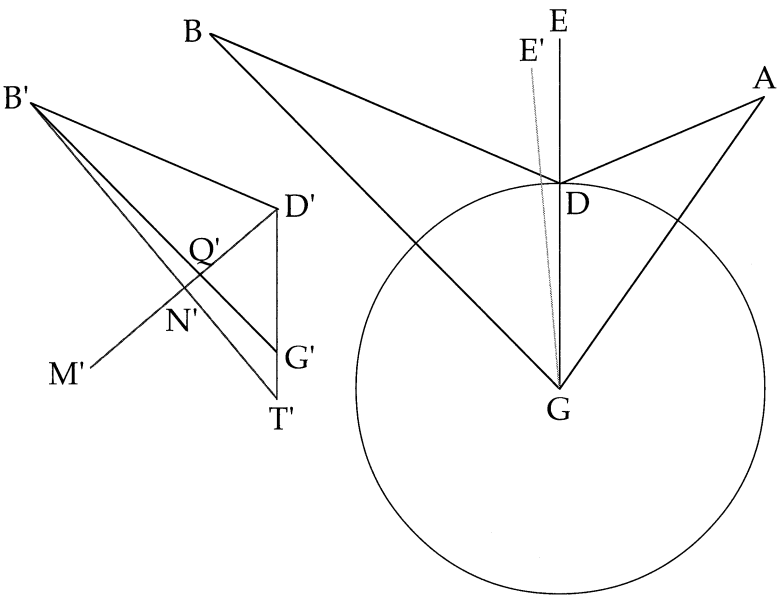


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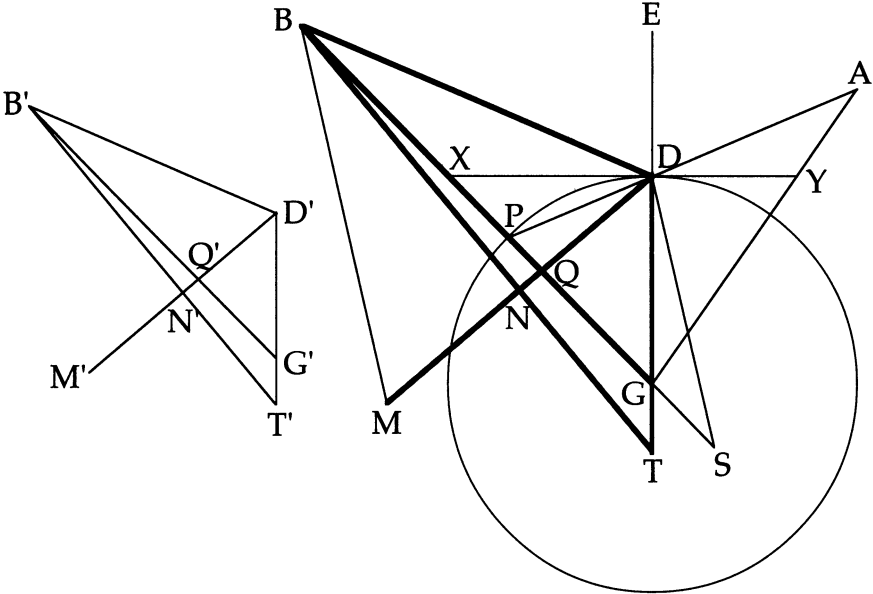


figure 17b

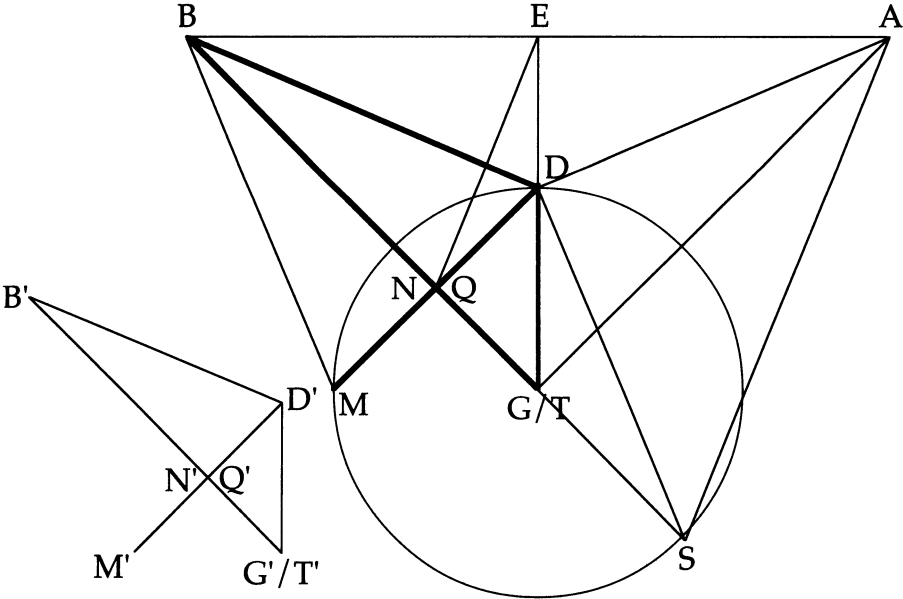


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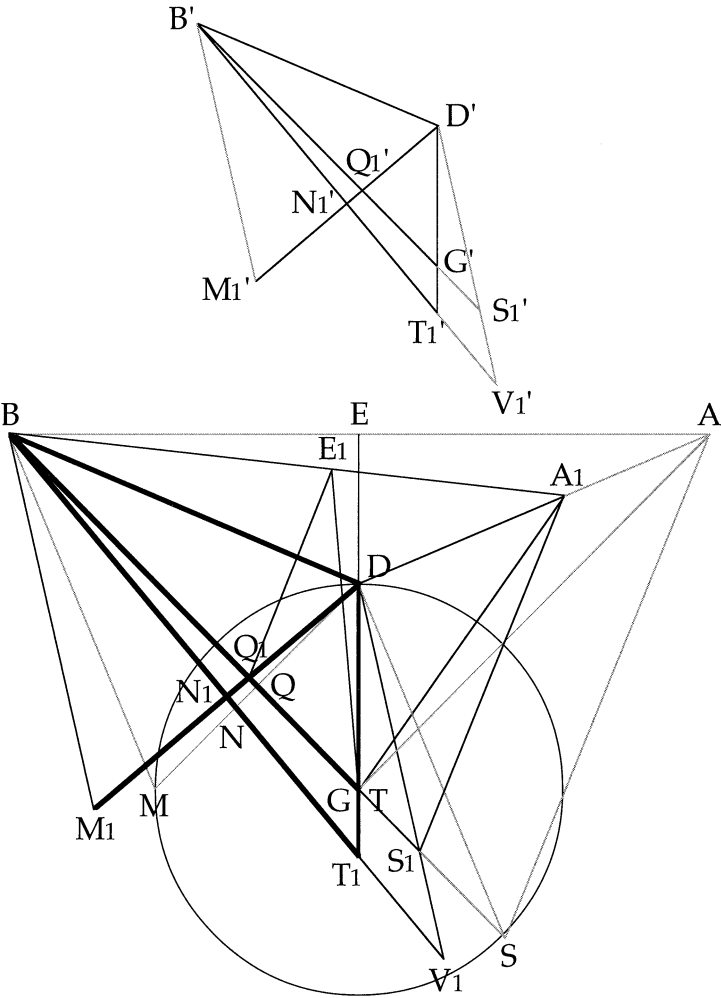


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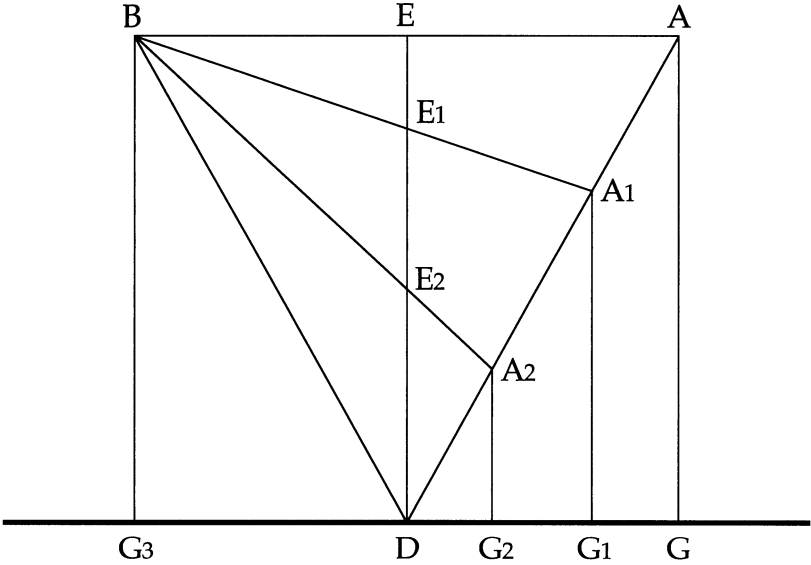


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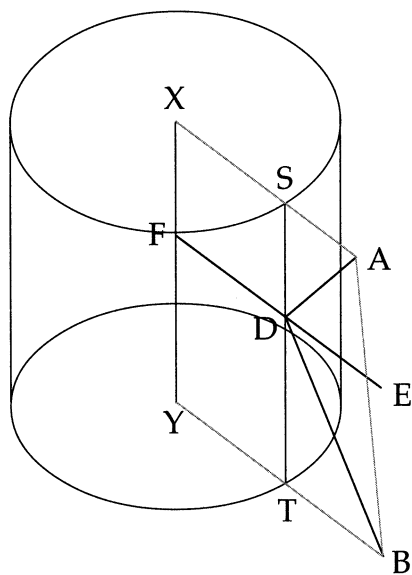


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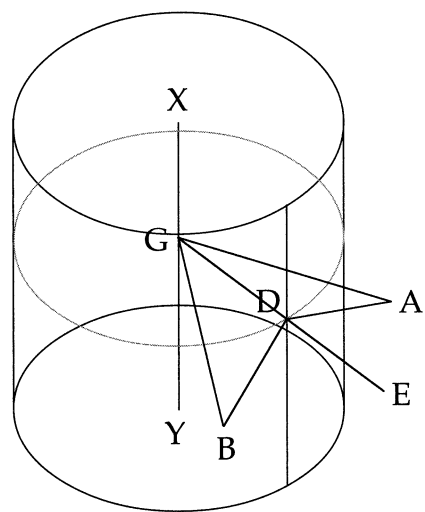


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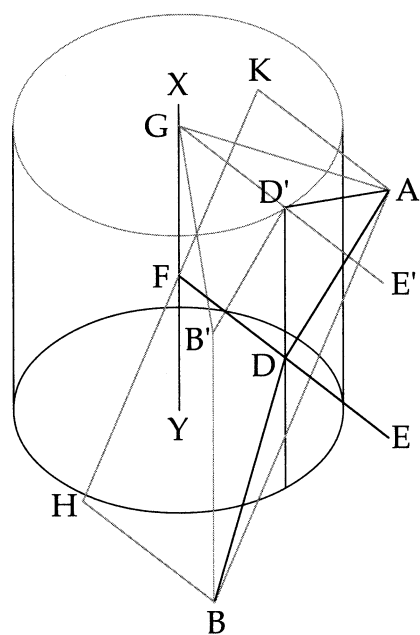


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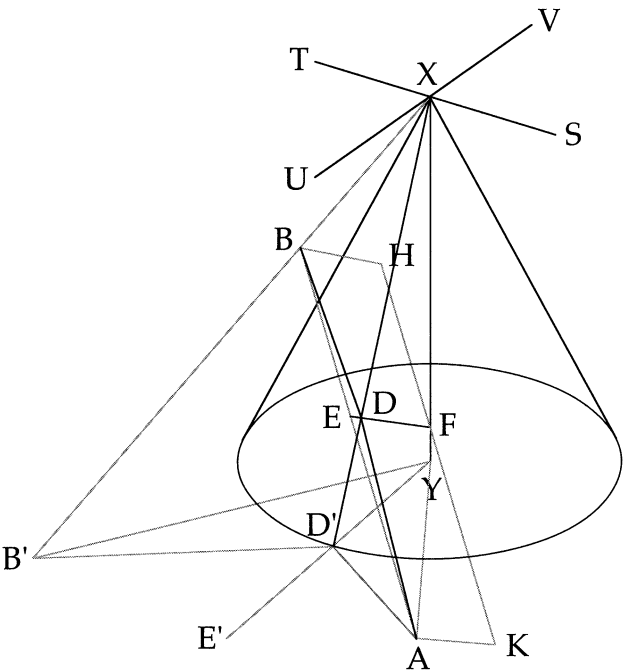


figure 19

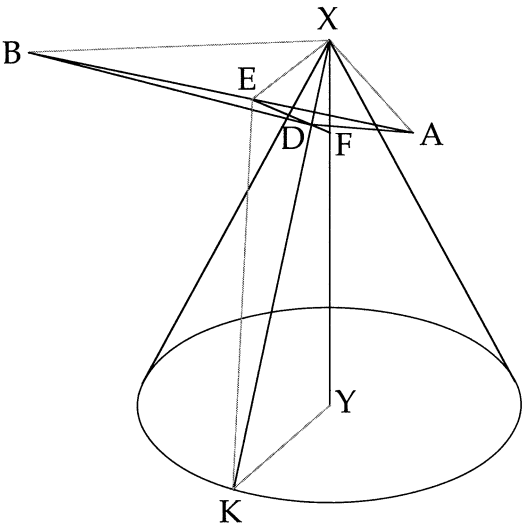


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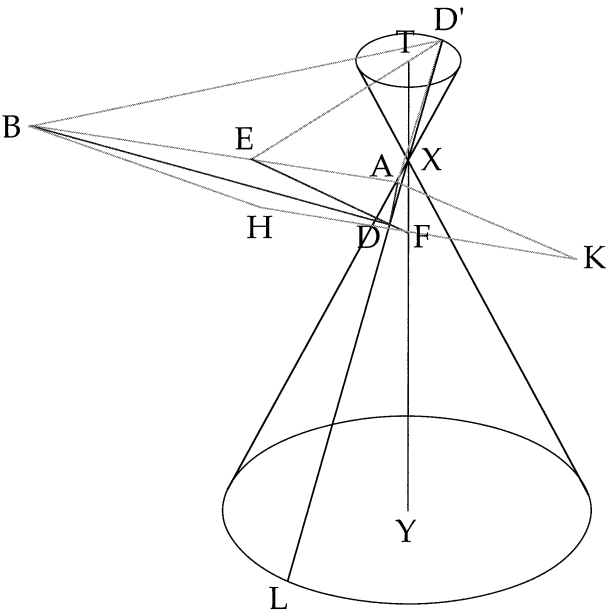


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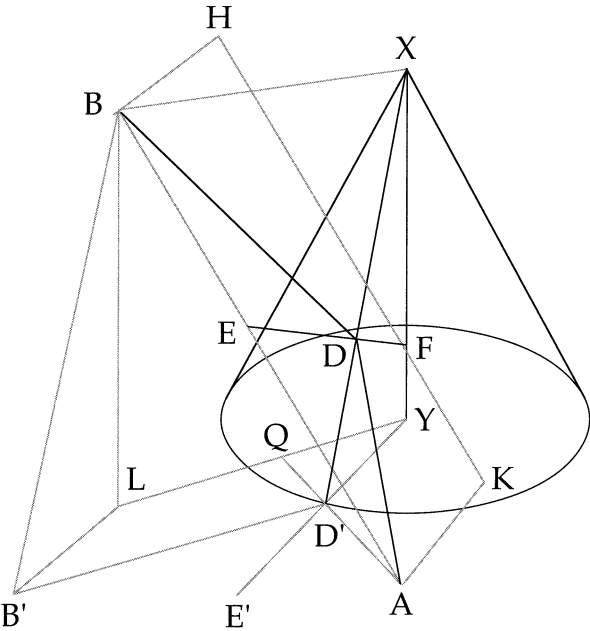


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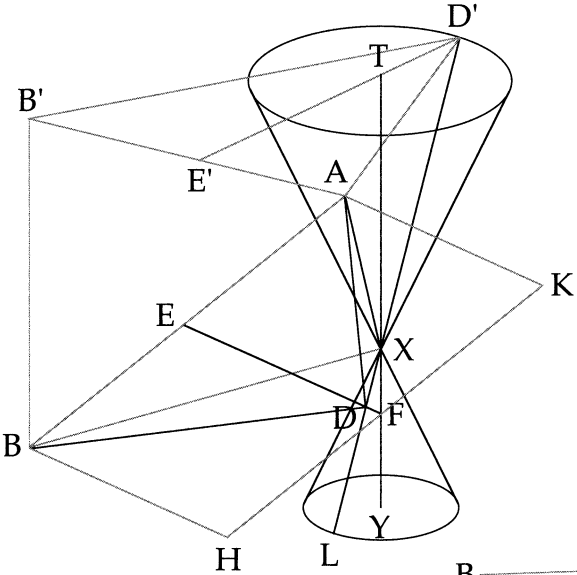


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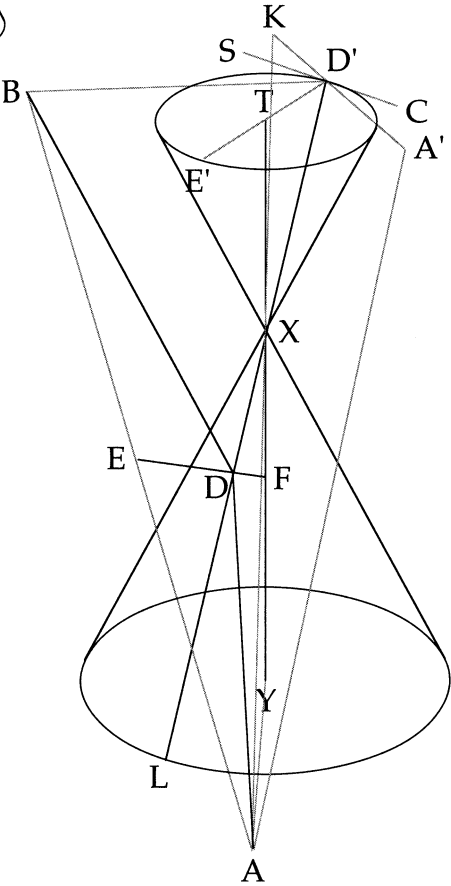


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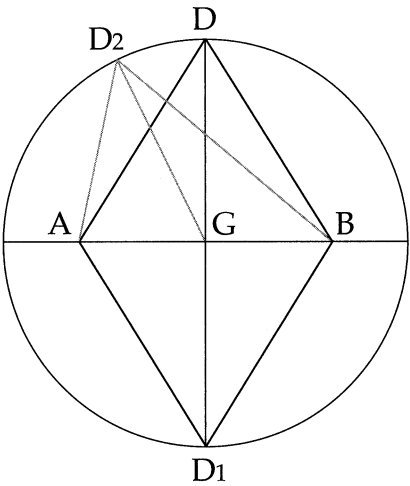


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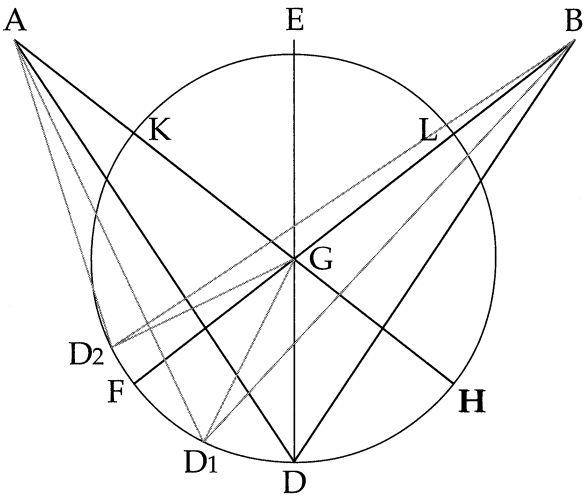


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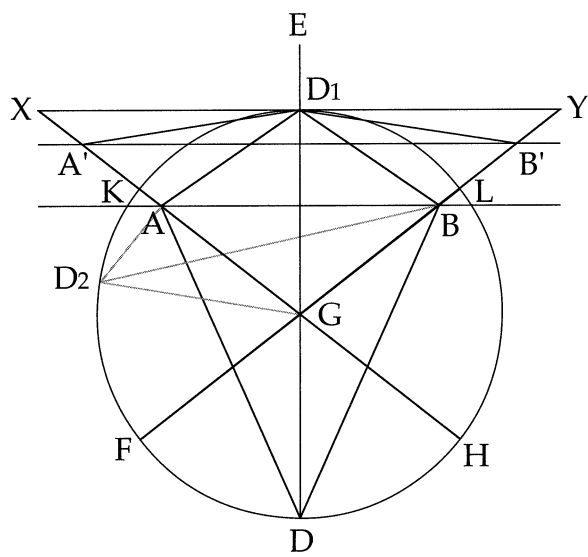


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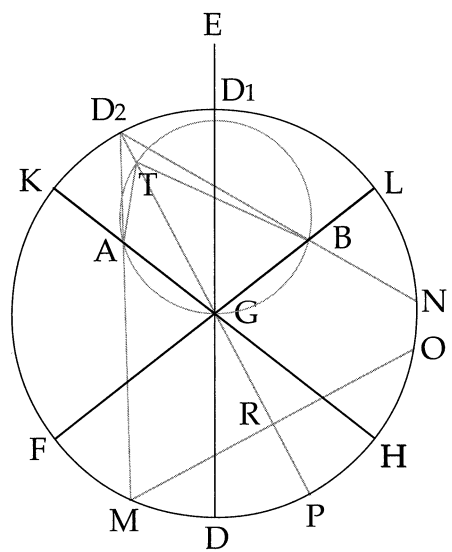


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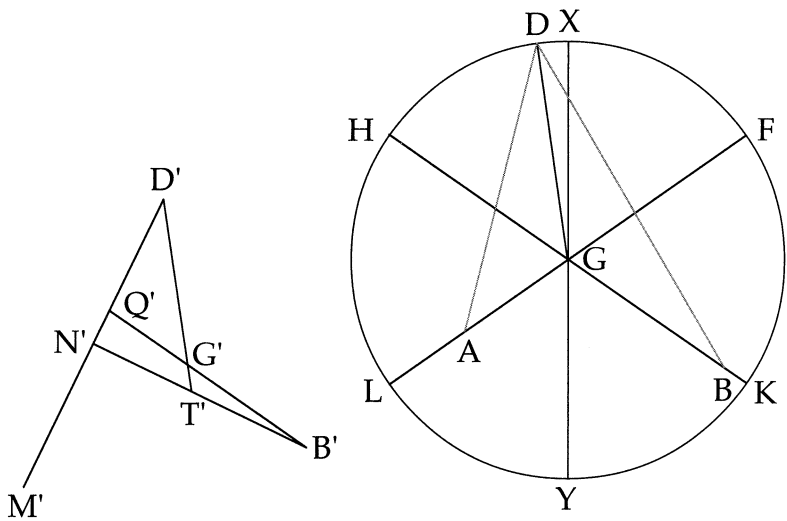


figure 20g

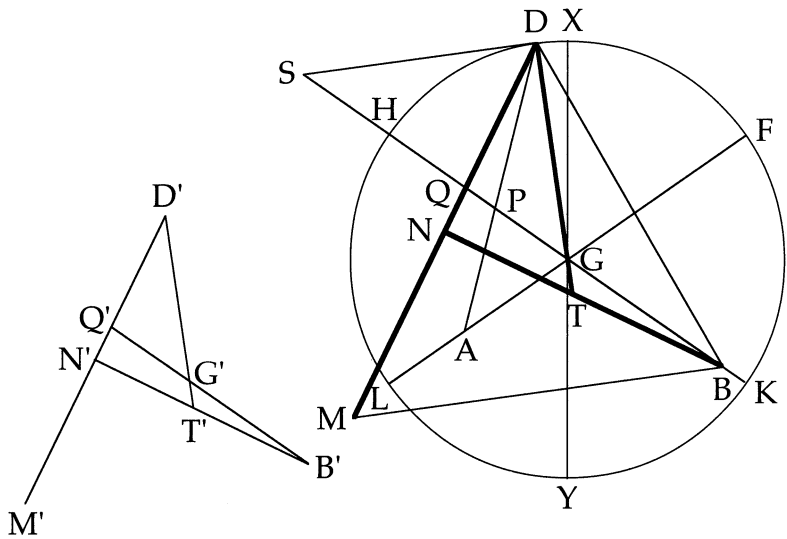


figure 20h

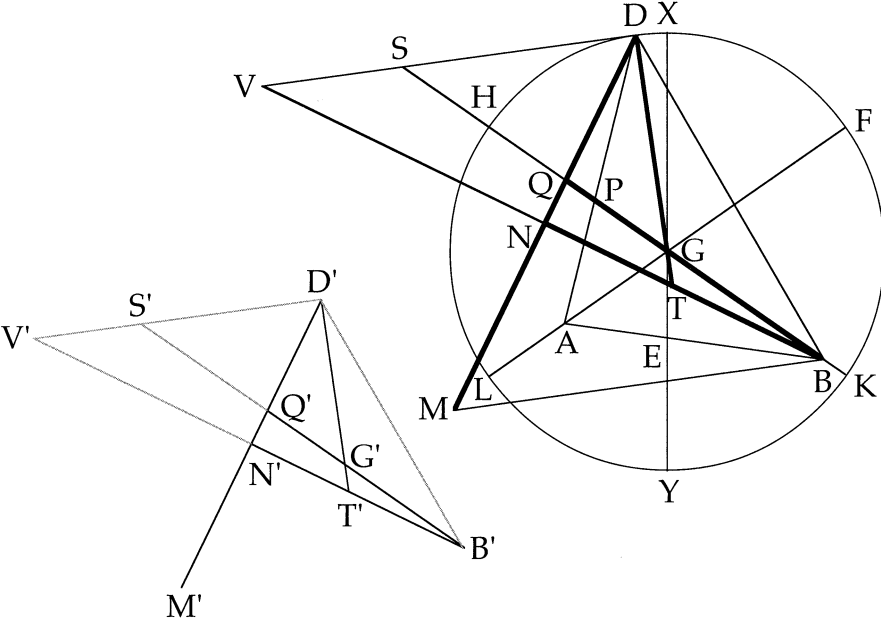


figure 20k

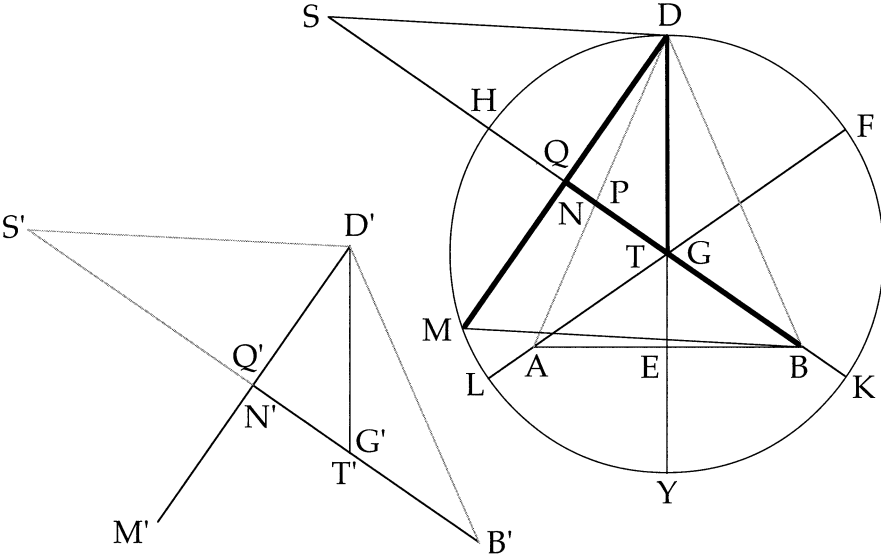


figure 20m

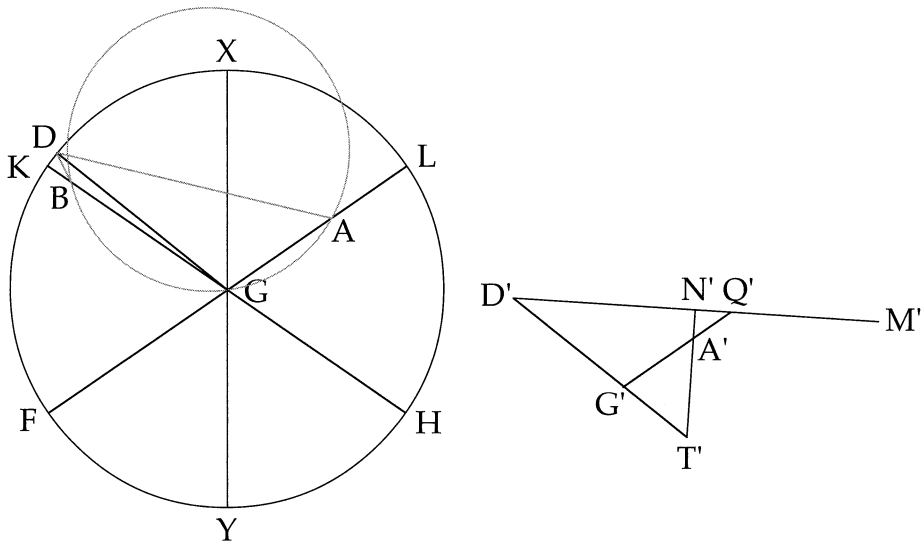


figure 20n

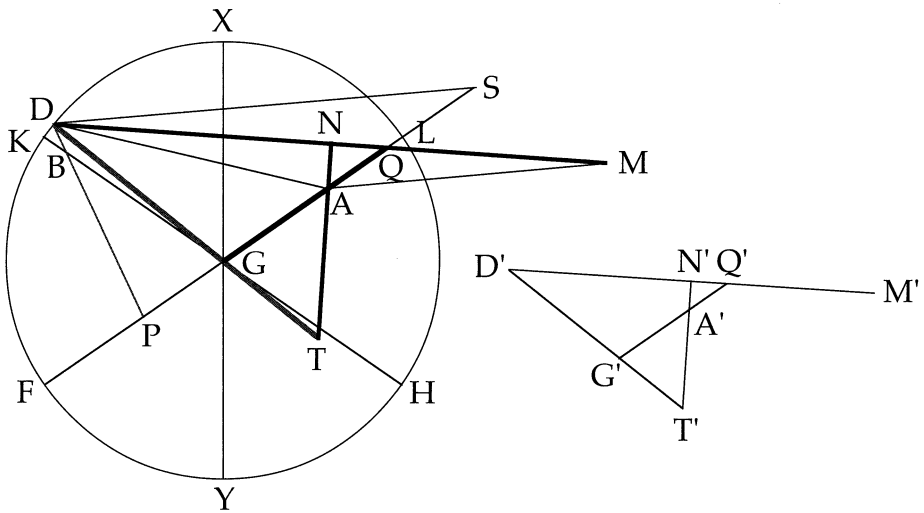


figure 20p

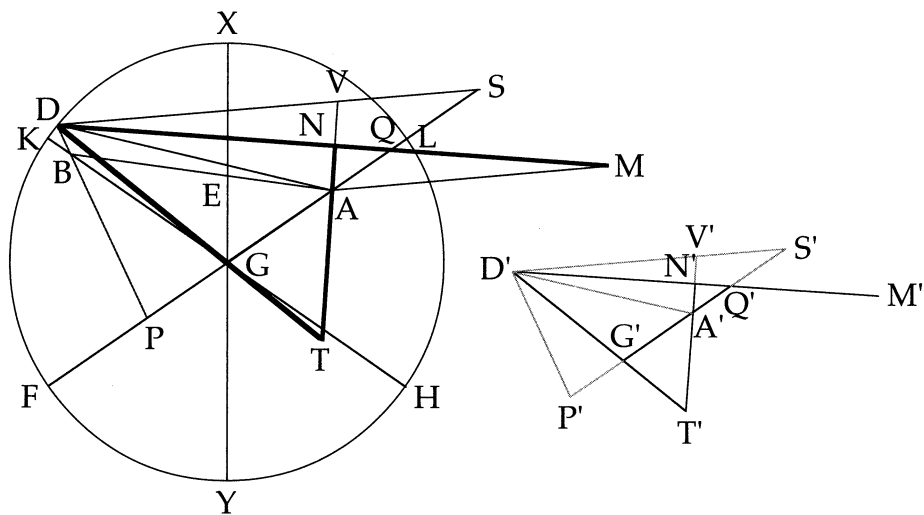


figure 20q

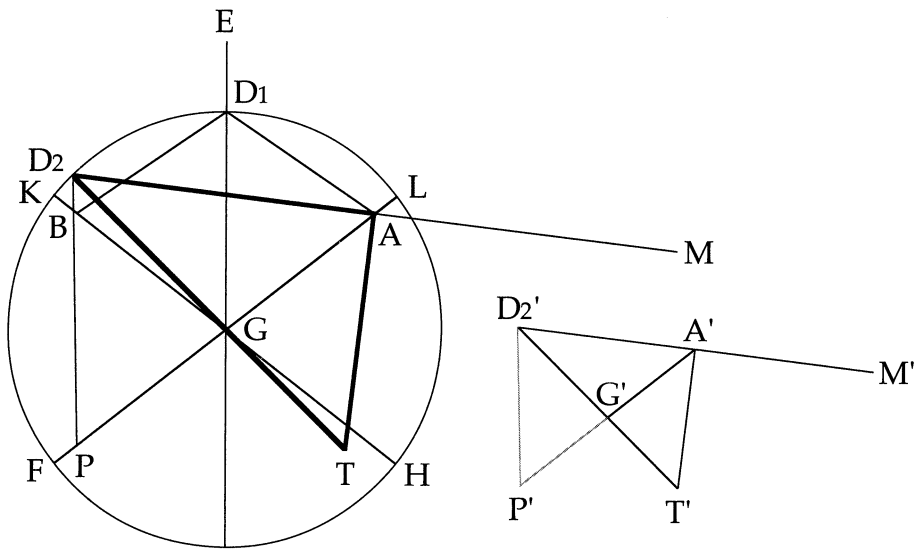


figure 20r

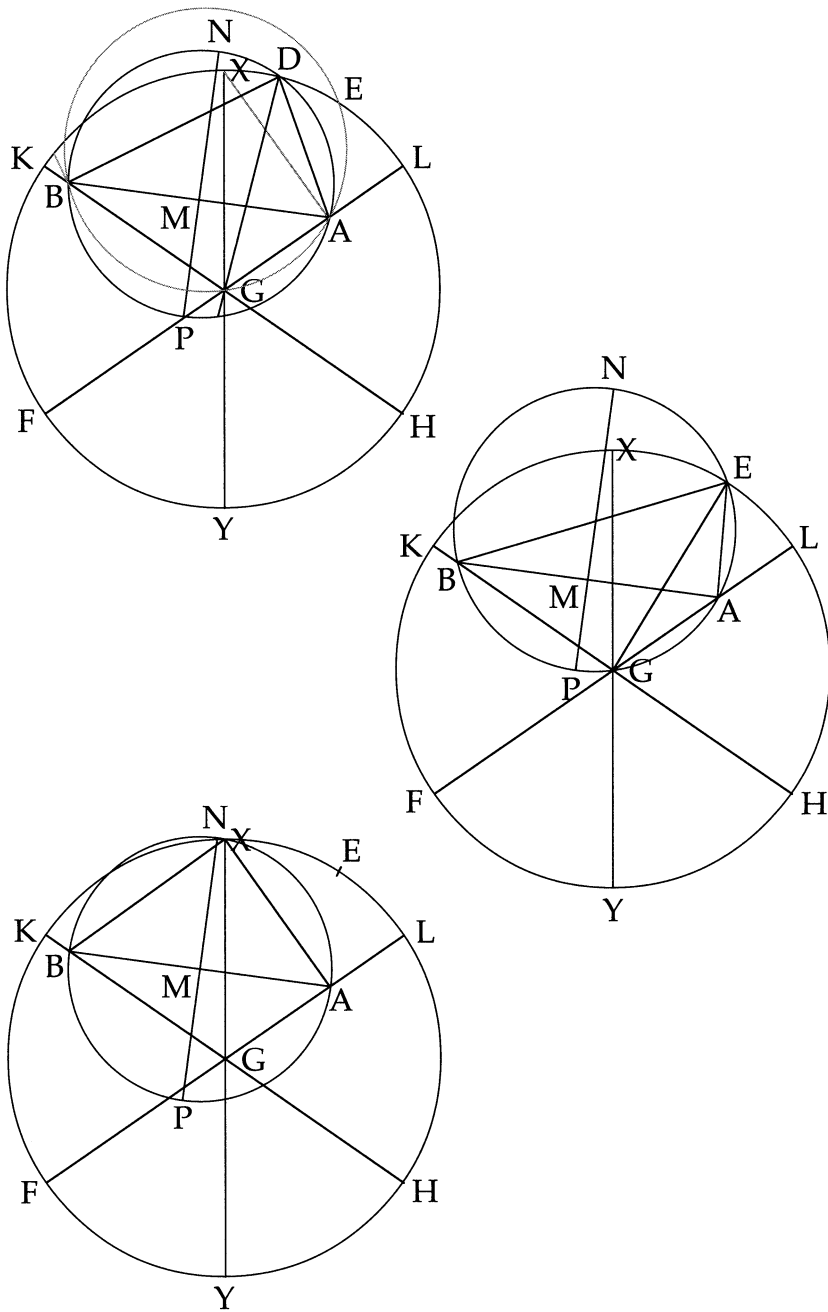


figure 20s

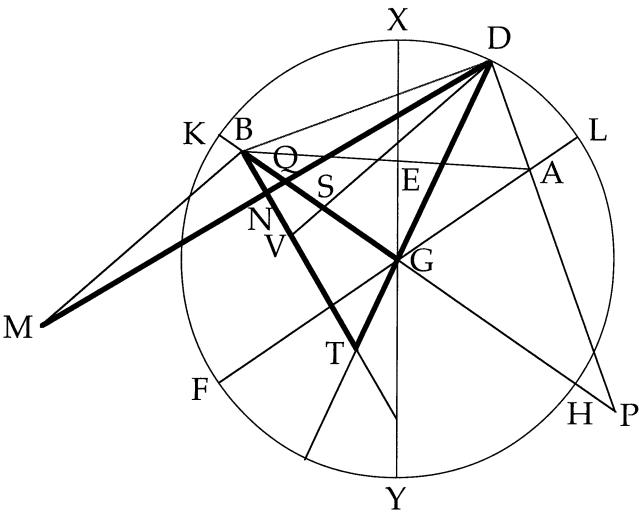
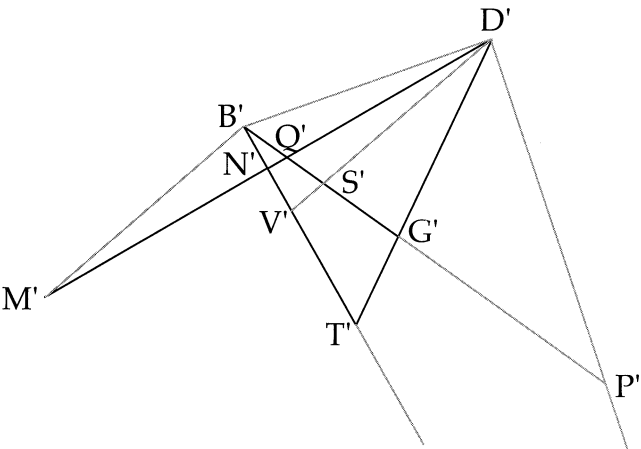


figure 20x

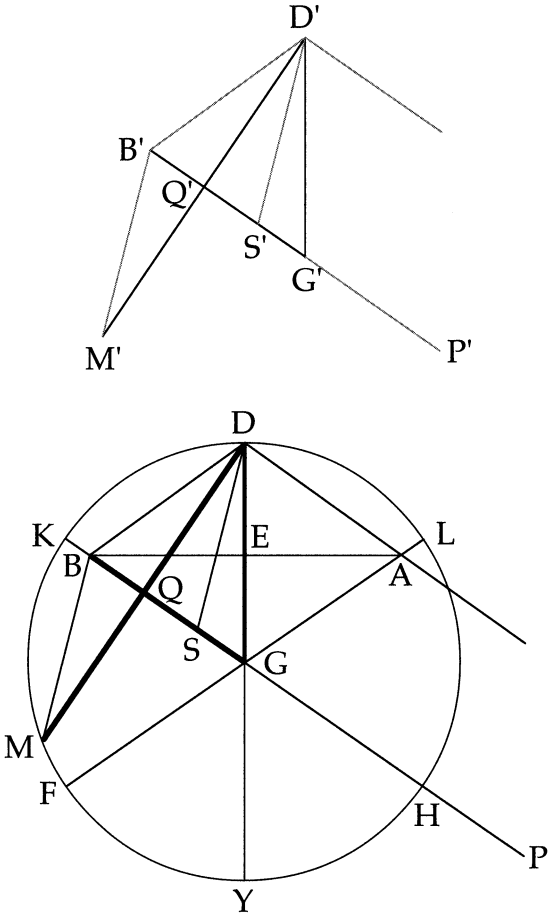


figure 20y

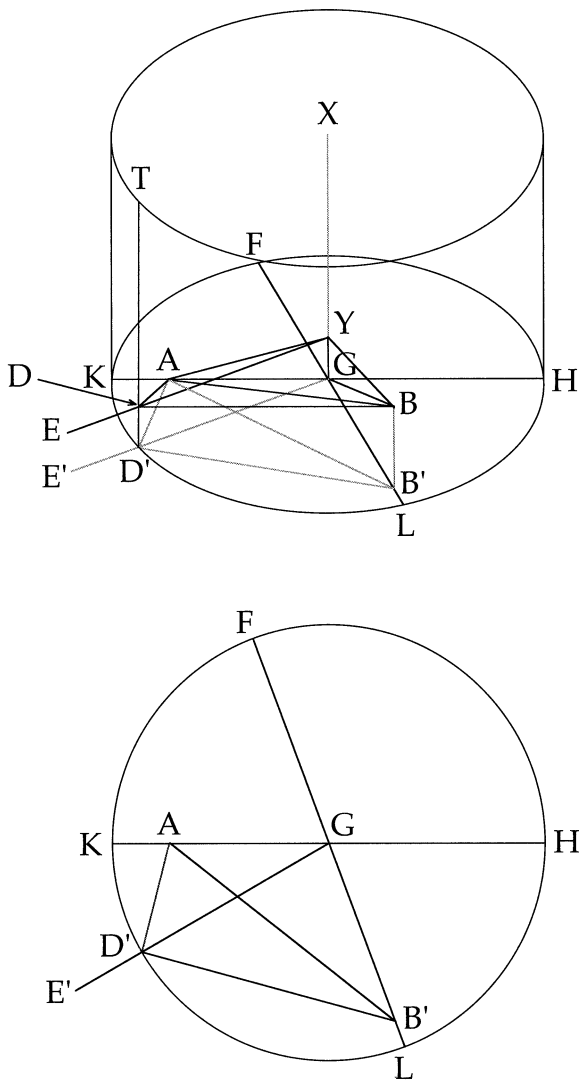


figure 21

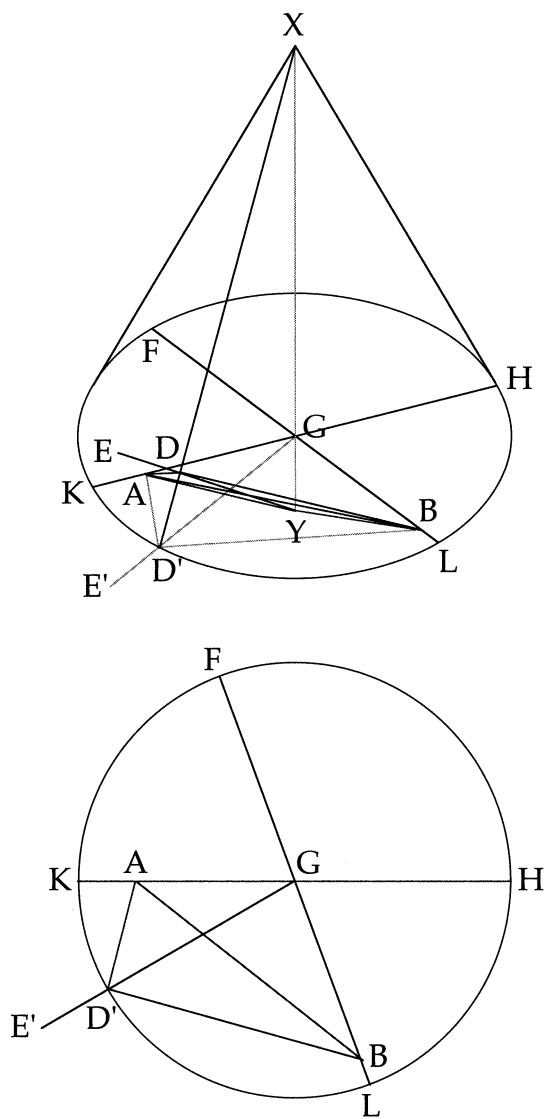


figure 22

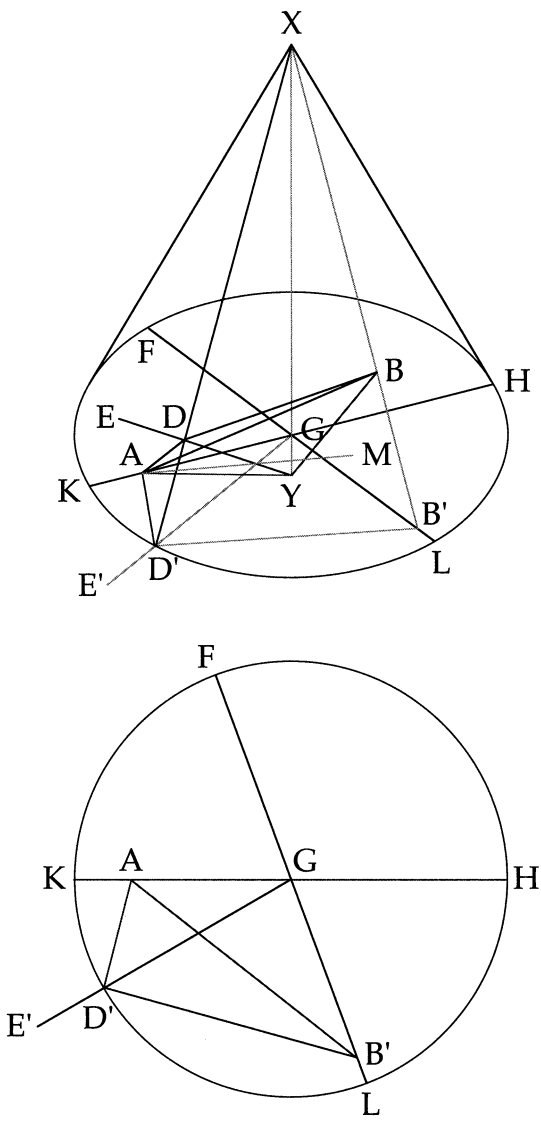


figure 22a

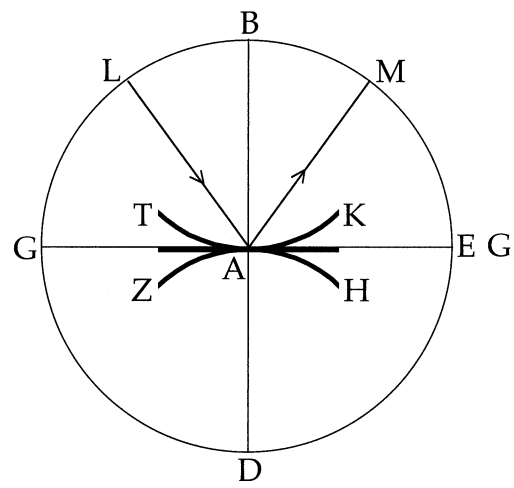


figure 23

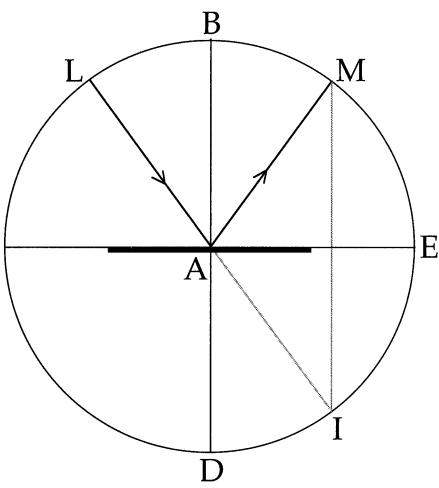


figure 23a

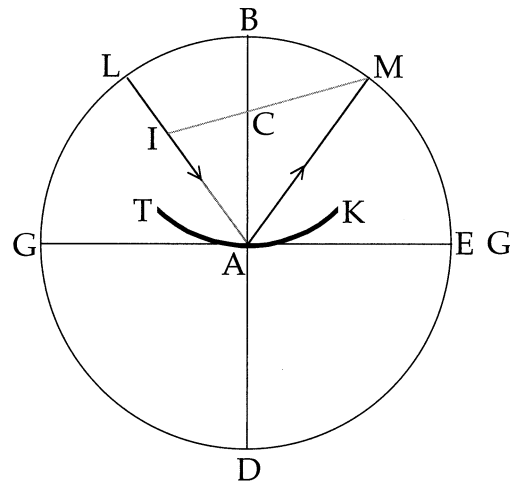


figure 23b

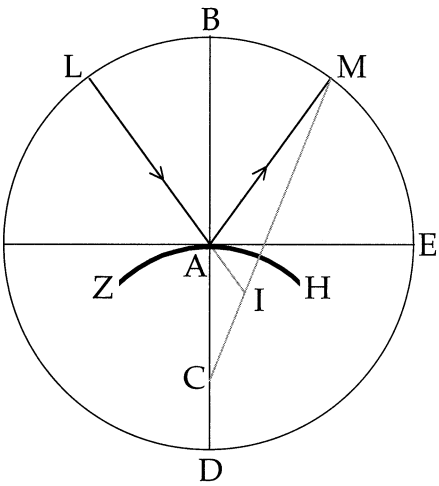


figure 23c

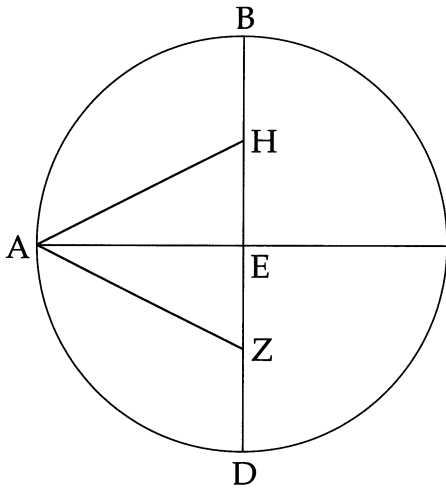


figure 24

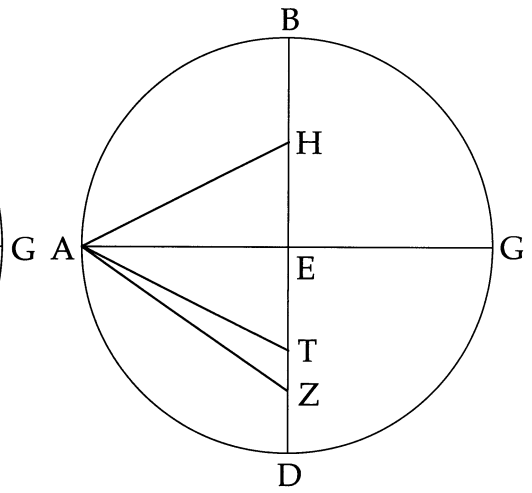


figure 24a

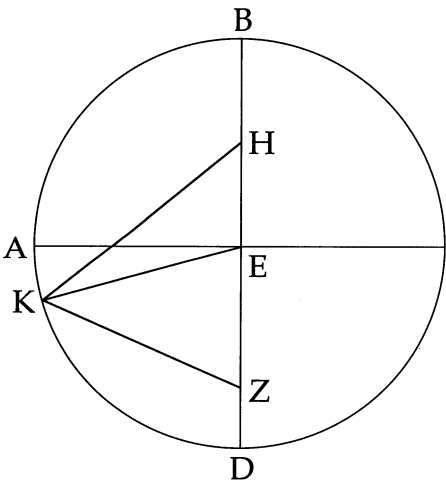


figure 24b

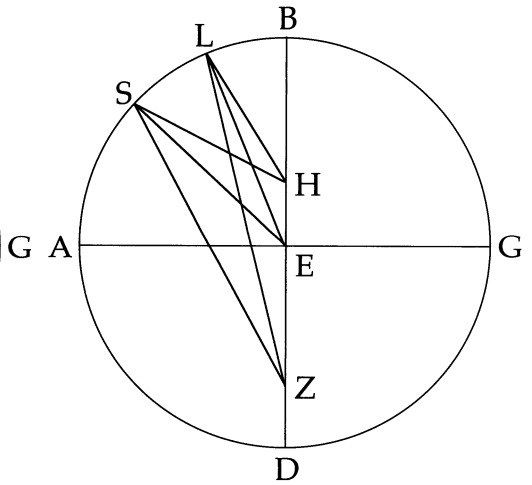


figure 24c

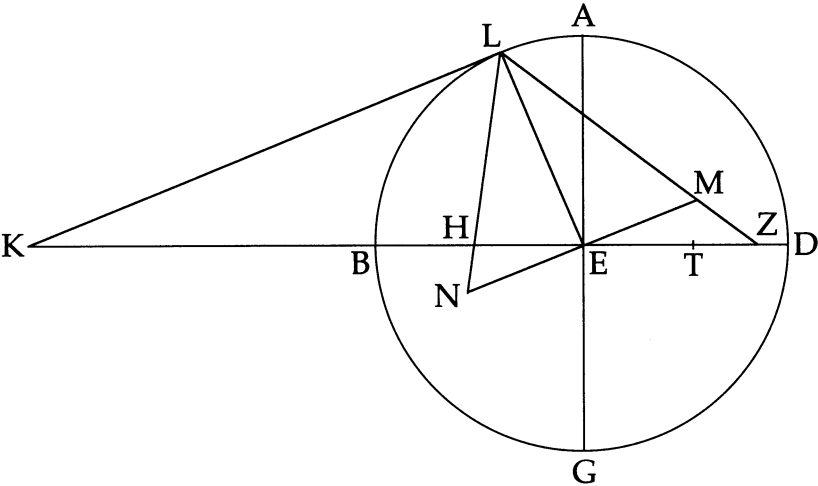


figure 25

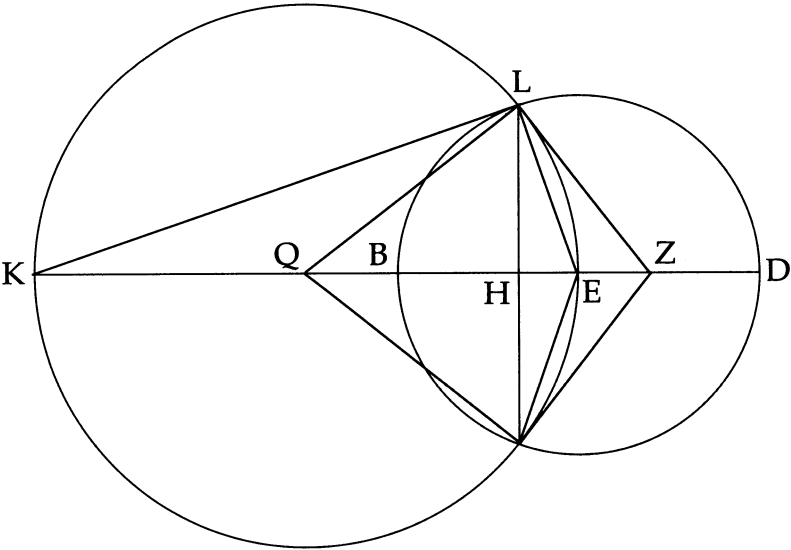


figure 25a

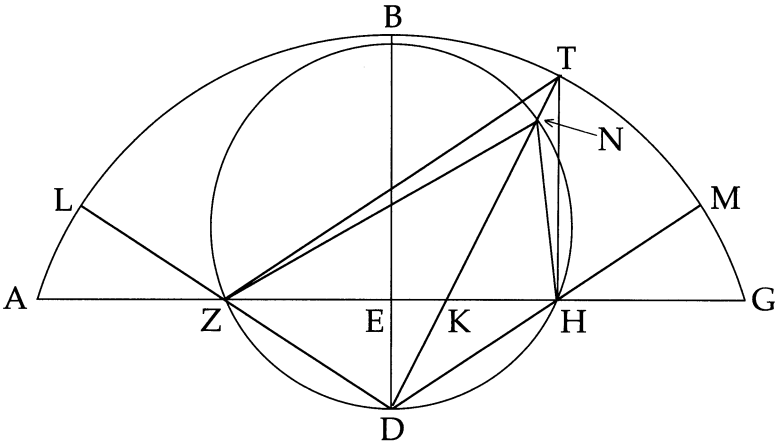


figure 26

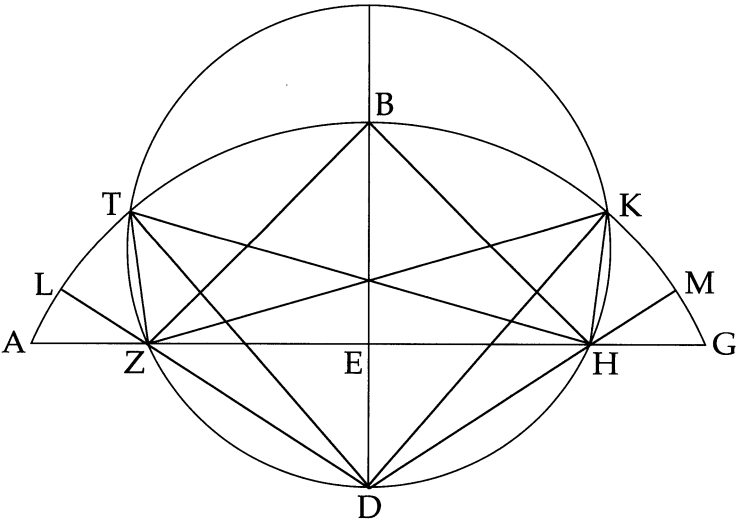


figure 26a

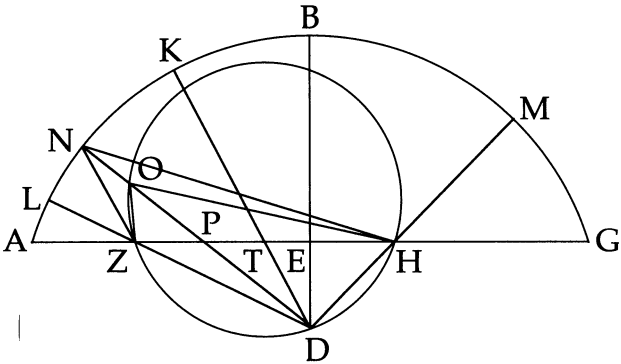


figure 27

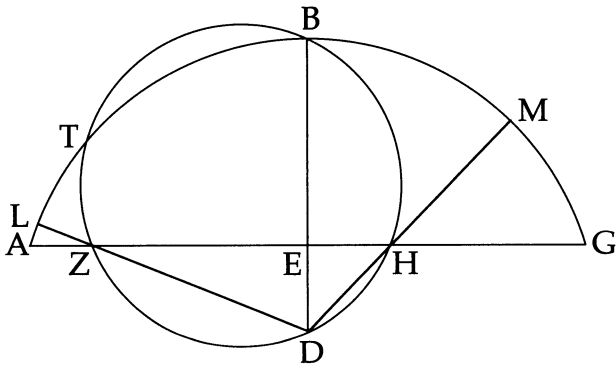


figure 27a

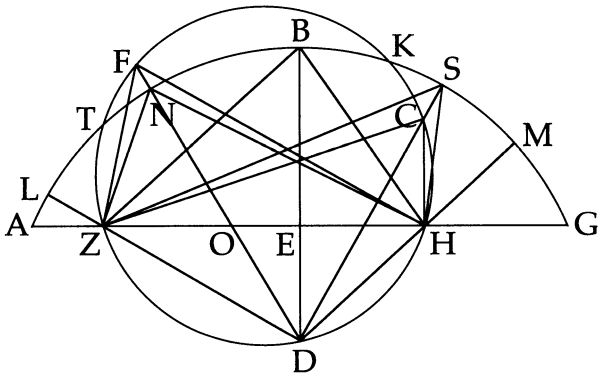


figure 27b

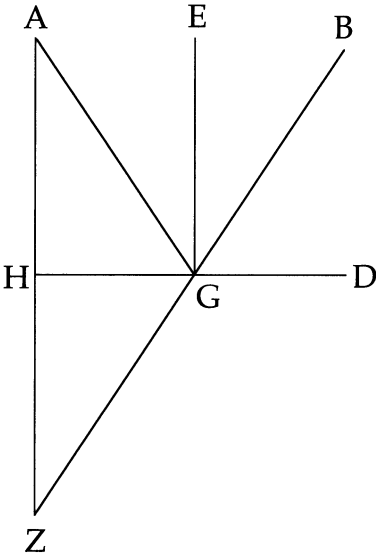


FIGURE 5.2.1

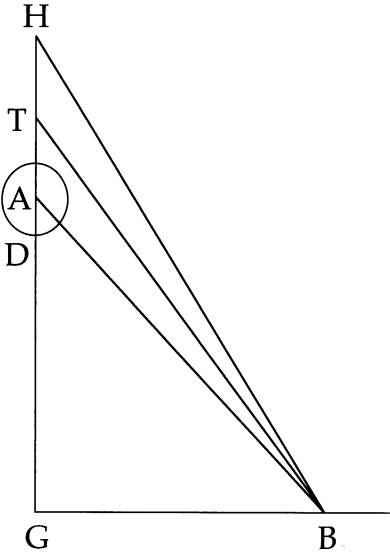


FIGURE 5.2.2

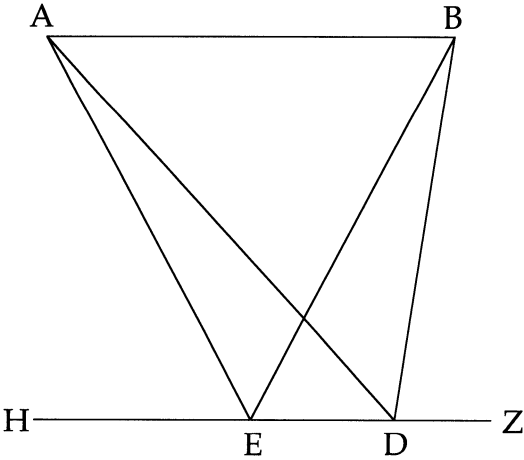


FIGURE 5.2.3

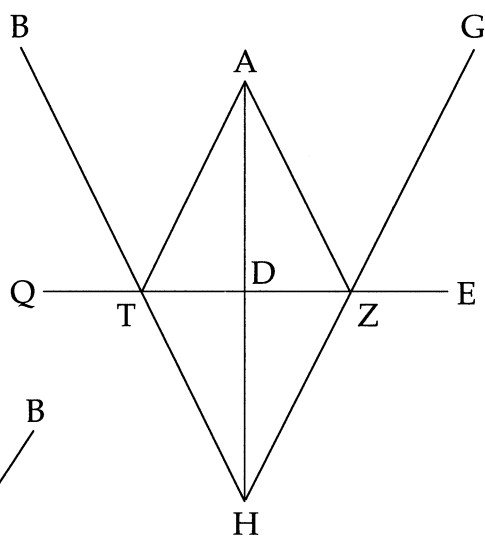


FIGURE 5.2.4

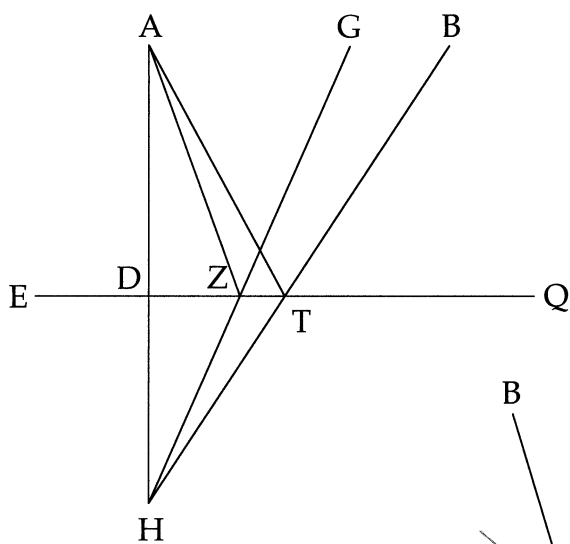


FIGURE 5.2.4a

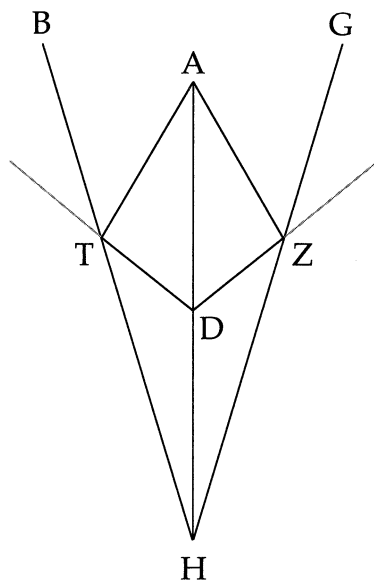


FIGURE 5.2.4b

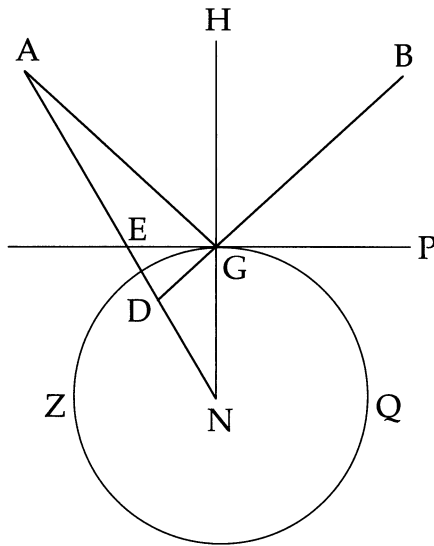


FIGURE 5.2.5

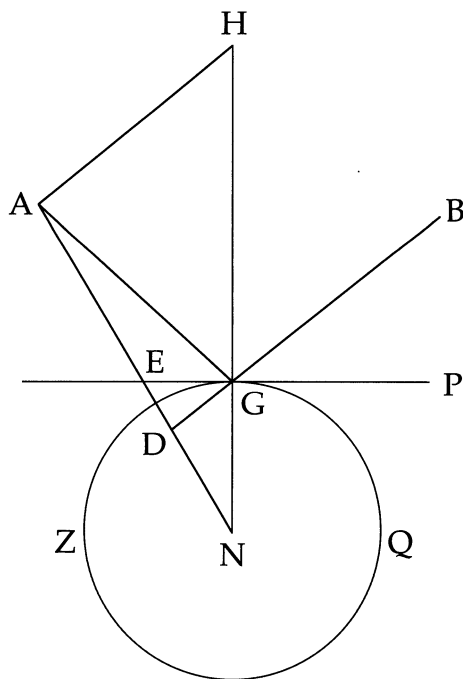


FIGURE 5.2.6

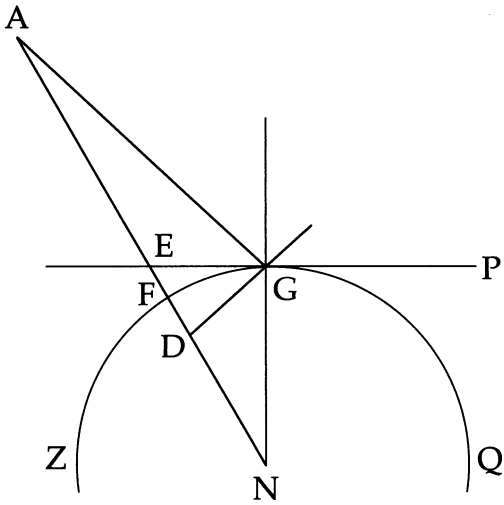


FIGURE 5.2.7

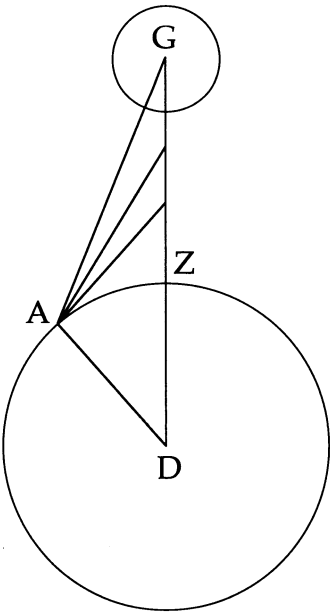


FIGURE 5.2.8

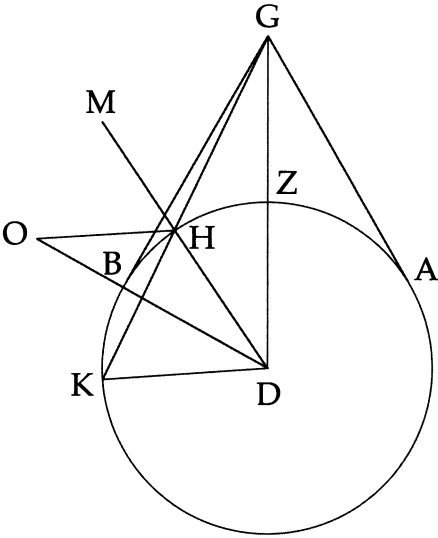


FIGURE 5.2.9

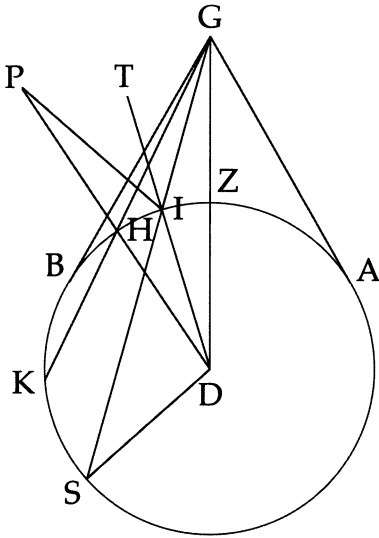


FIGURE 5.2.9a

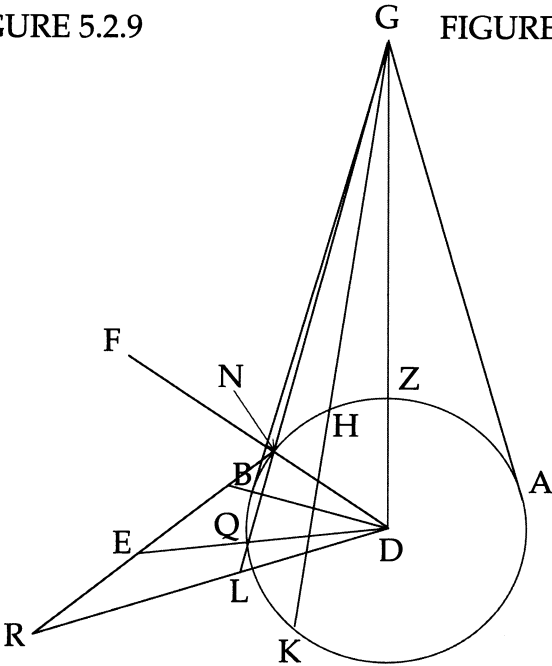


FIGURE 5.2.9b

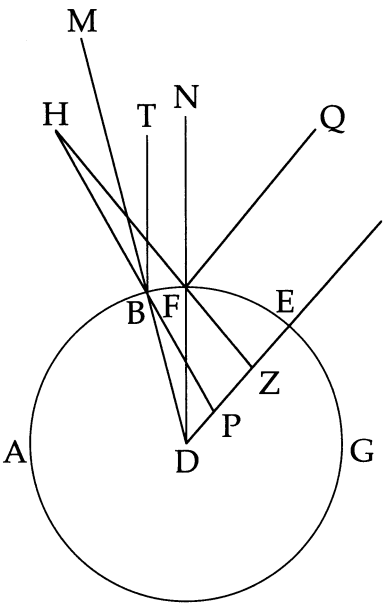


FIGURE 5.2.10

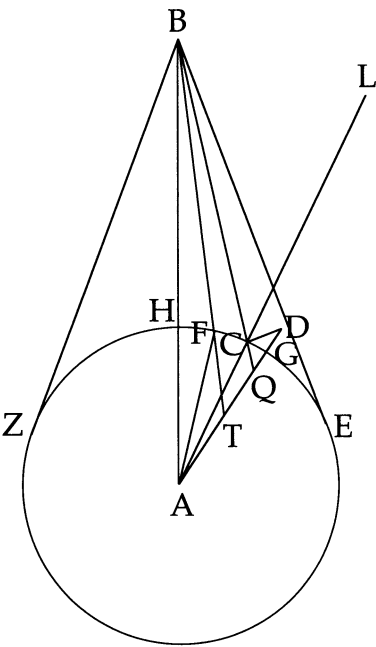


FIGURE 5.2.11

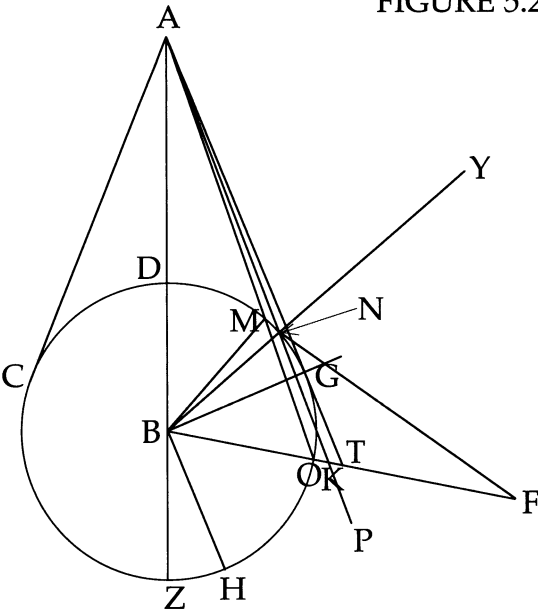


FIGURE 5.2.12

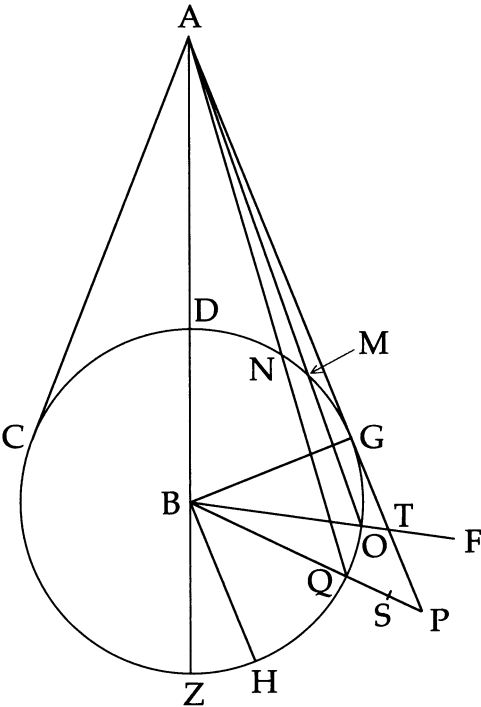


FIGURE 5.2.14

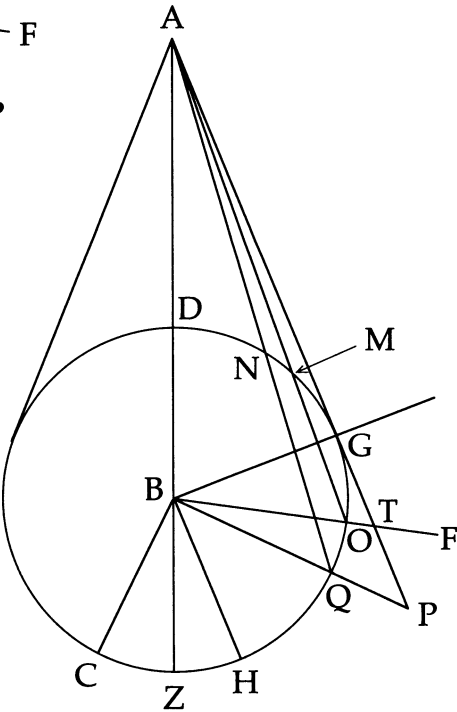


FIGURE 5.2.15

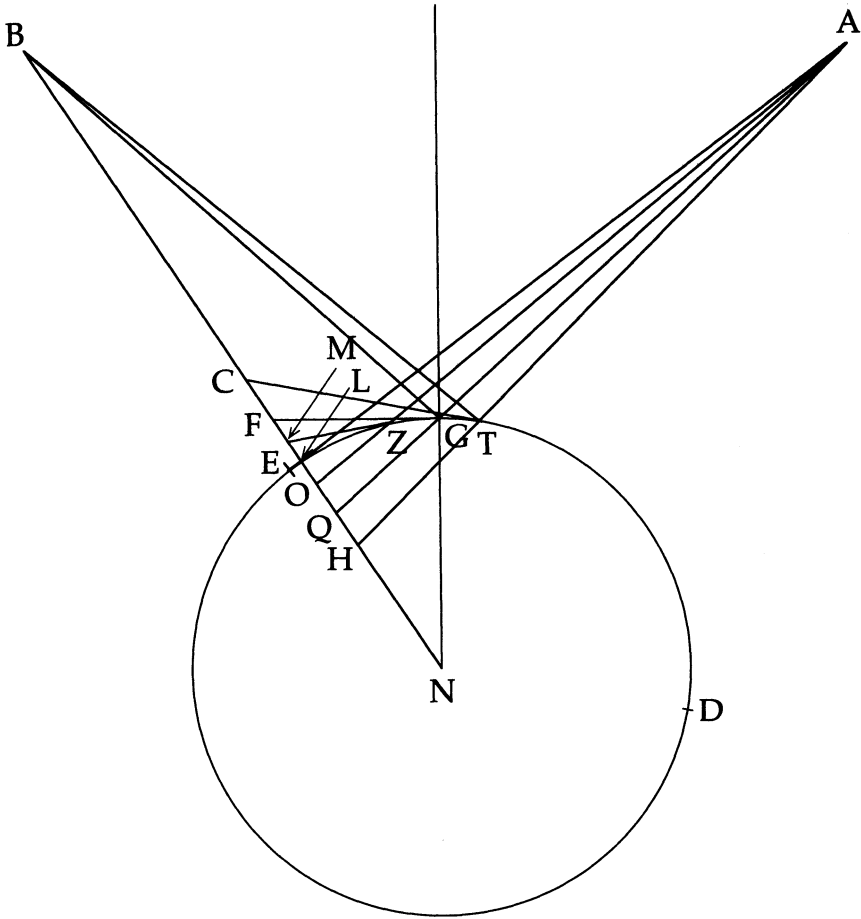


FIGURE 5.2.16

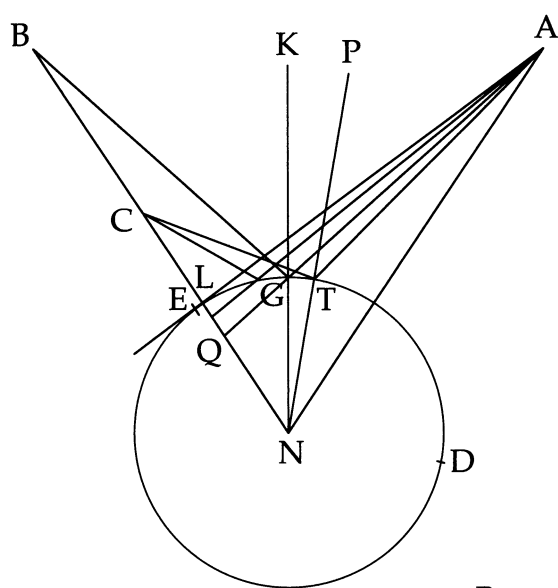


FIGURE 5.2.17

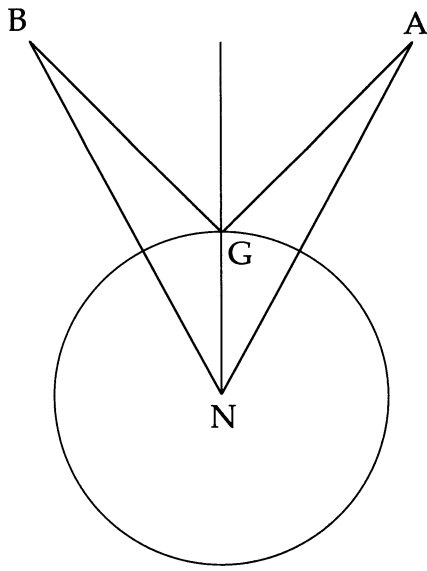


FIGURE 5.2.18

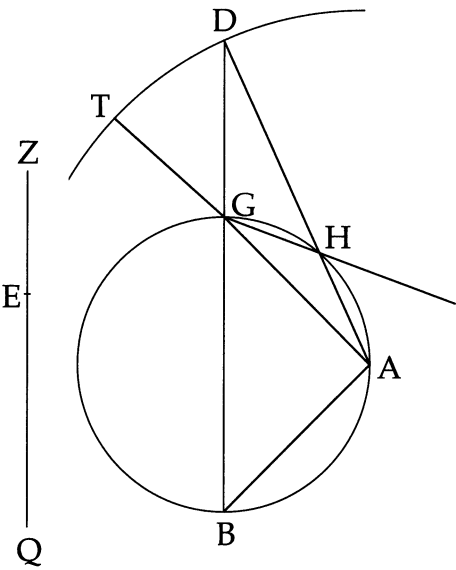


FIGURE 5.2.19a

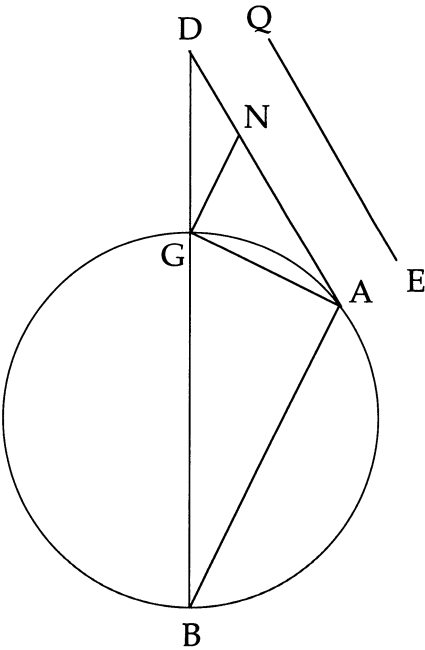
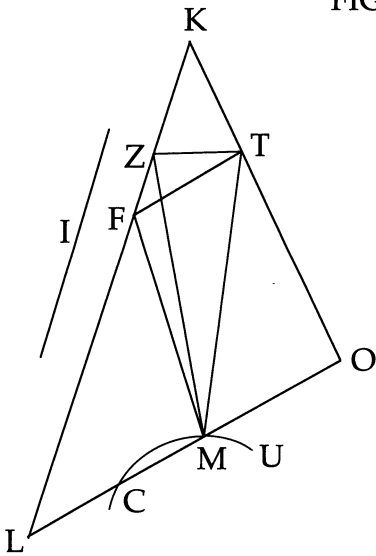


FIGURE 5.2.19c

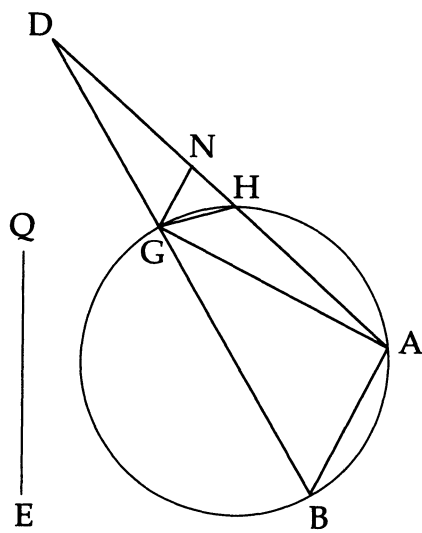


FIGURE 5.2.19d

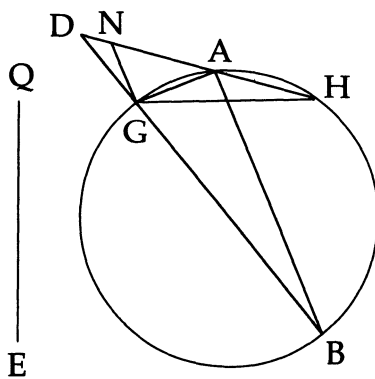


FIGURE 5.2.19e

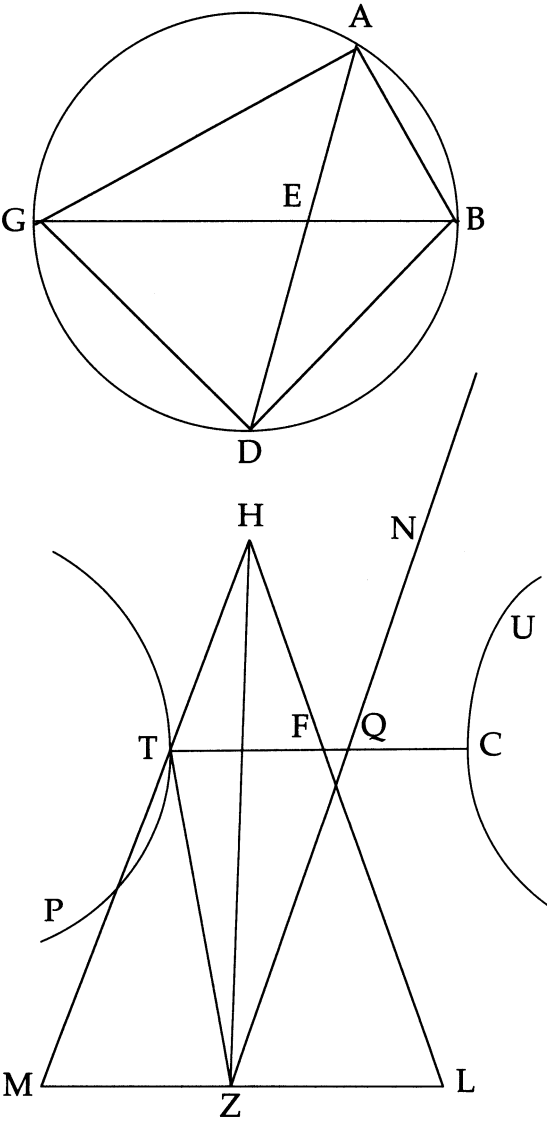


FIGURE 5.2.20

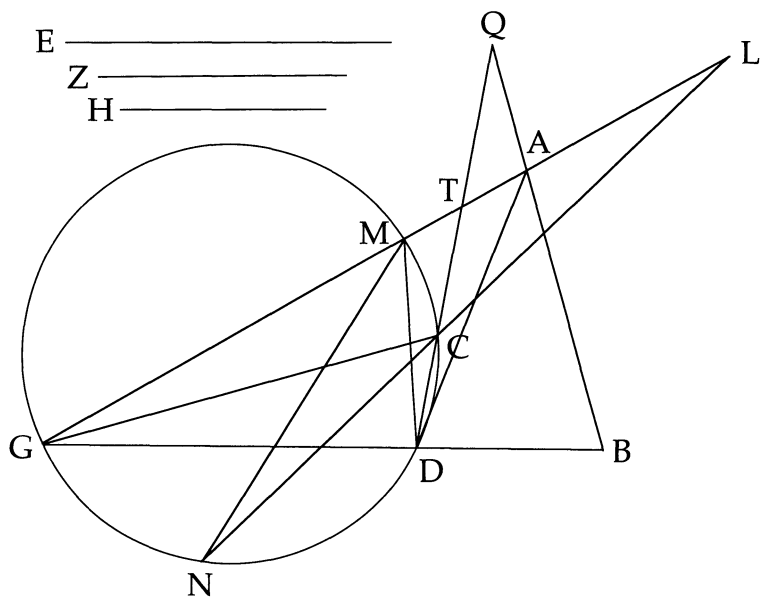


FIGURE 5.2.21

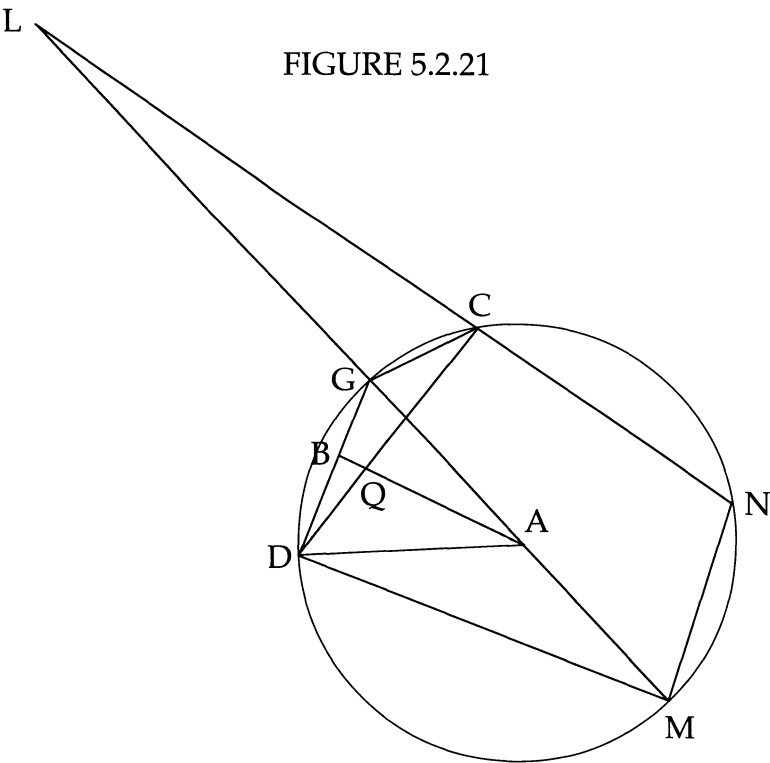


FIGURE 5.2.21a

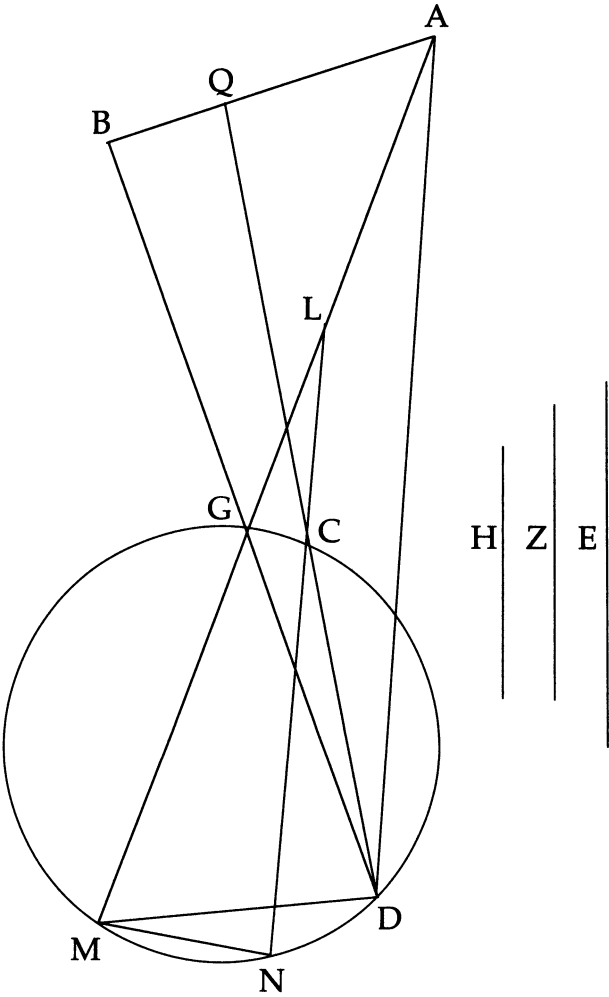


FIGURE 5.2.21b

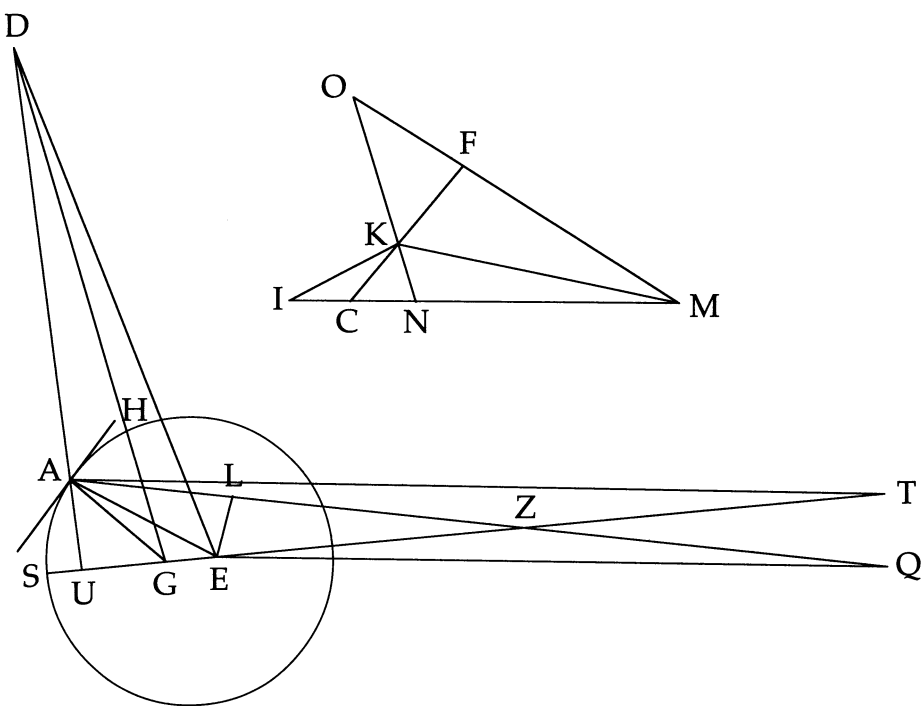


FIGURE 5.2.22

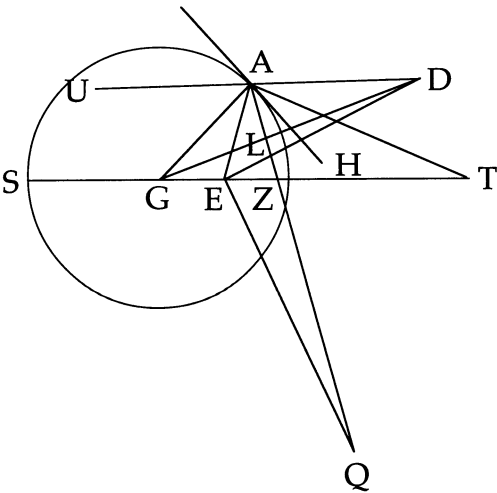


FIGURE 5.2.22a

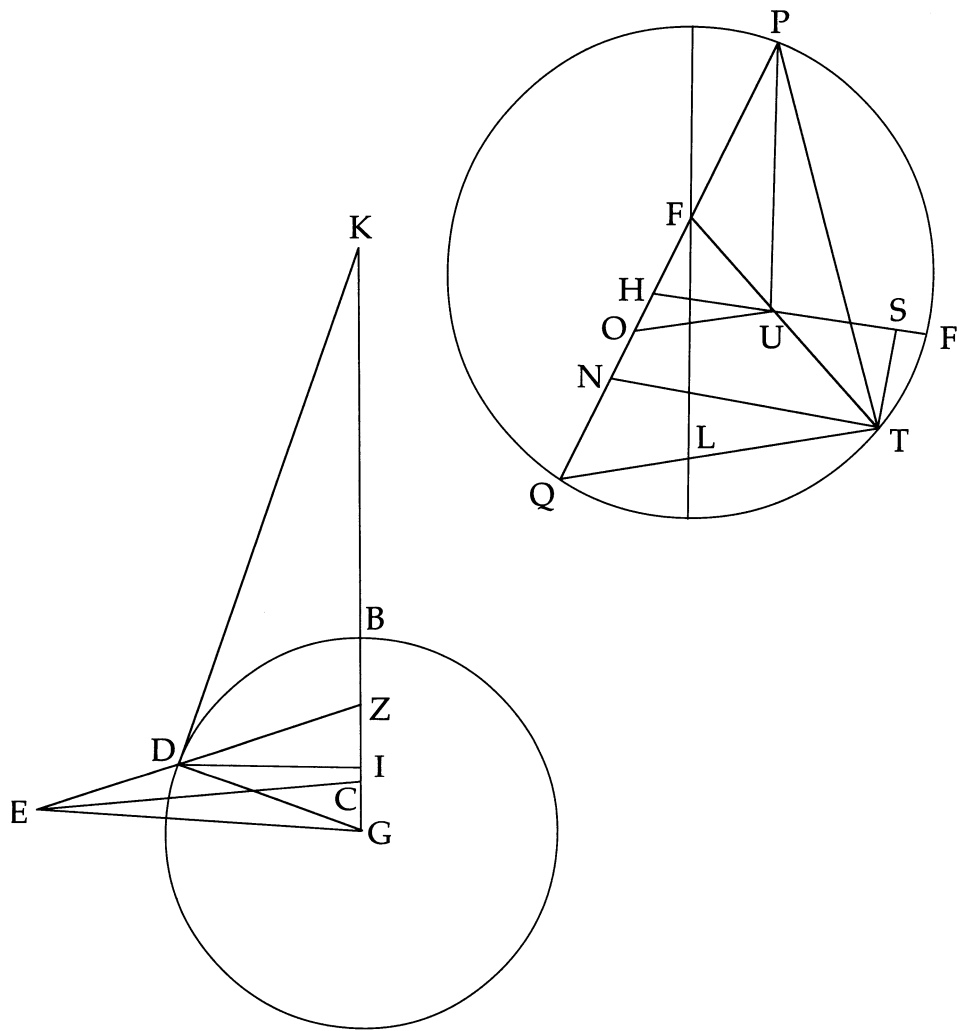


FIGURE 5.2.23

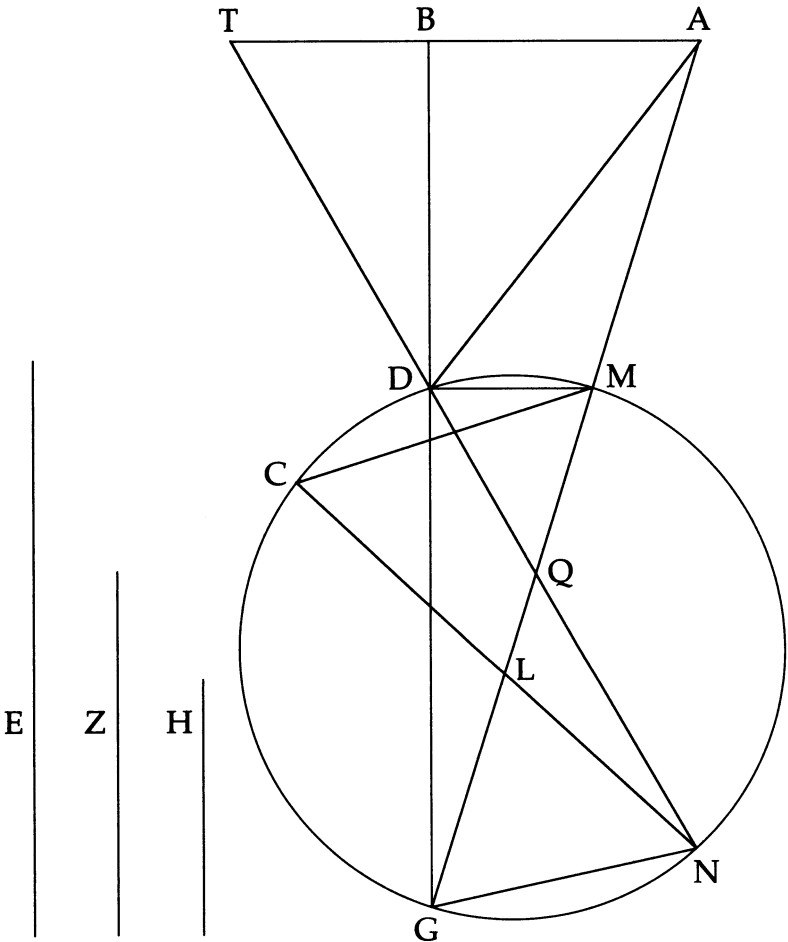


FIGURE 5.2.24

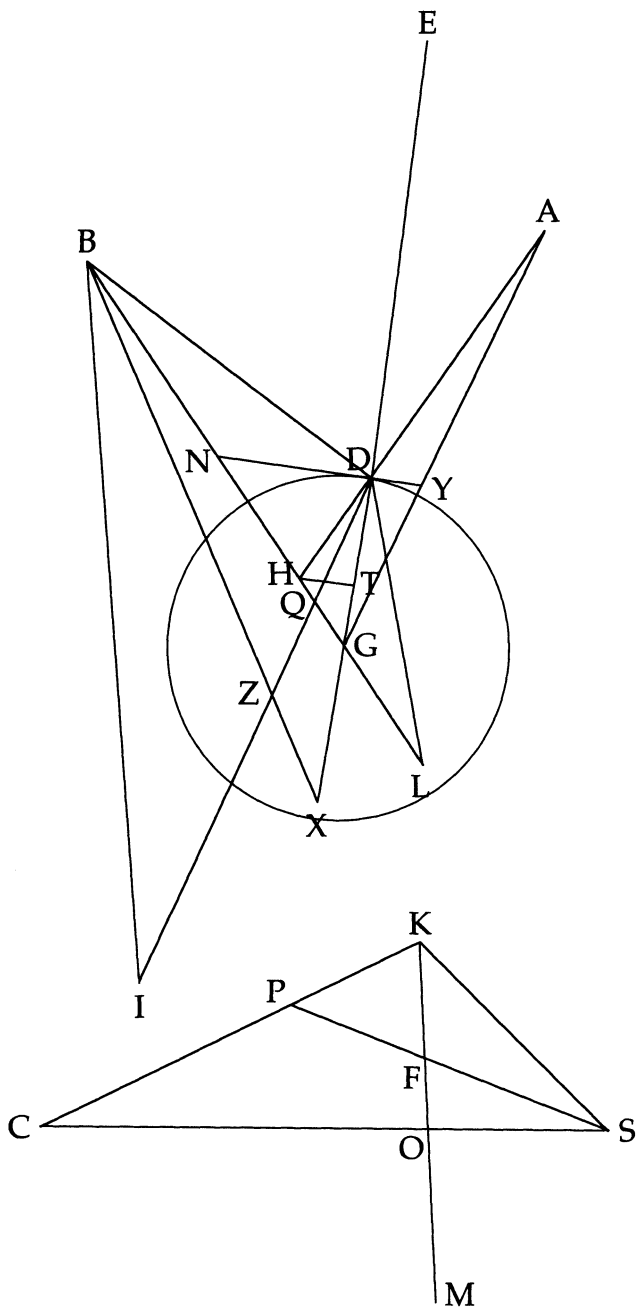


FIGURE 5.2.25

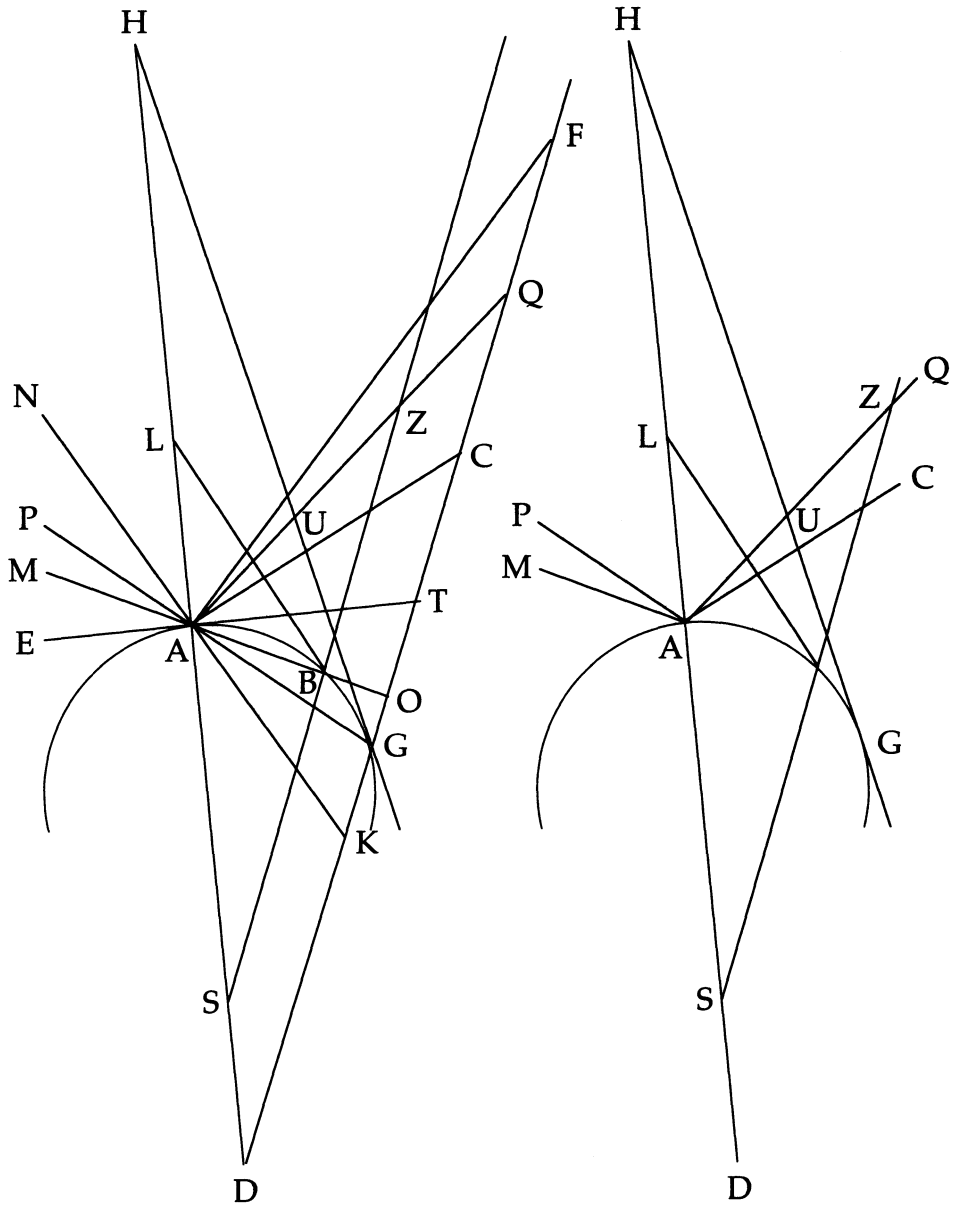


FIGURE 5.2.27

FIGURE 5.2.27a

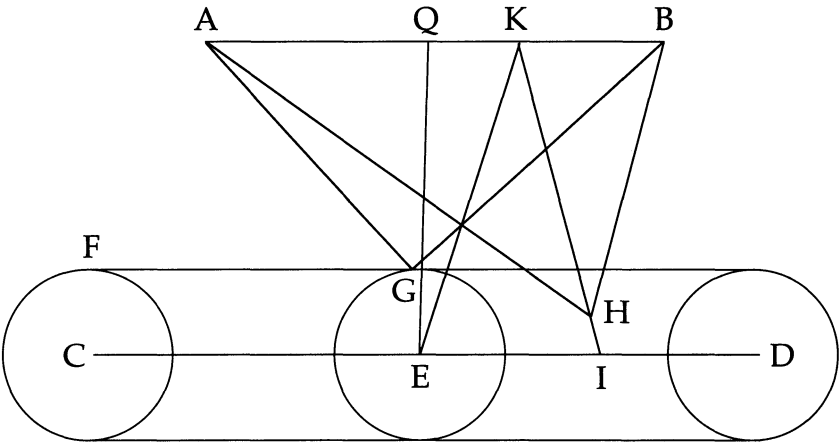


FIGURE 5.2.28

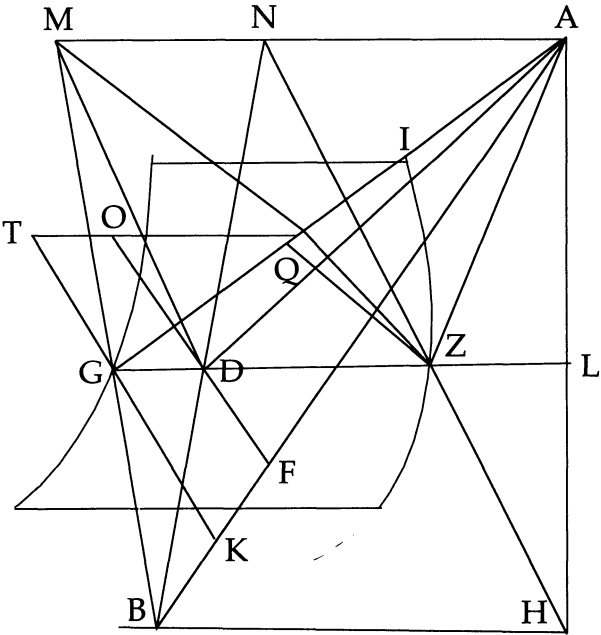


FIGURE 5.2.28a

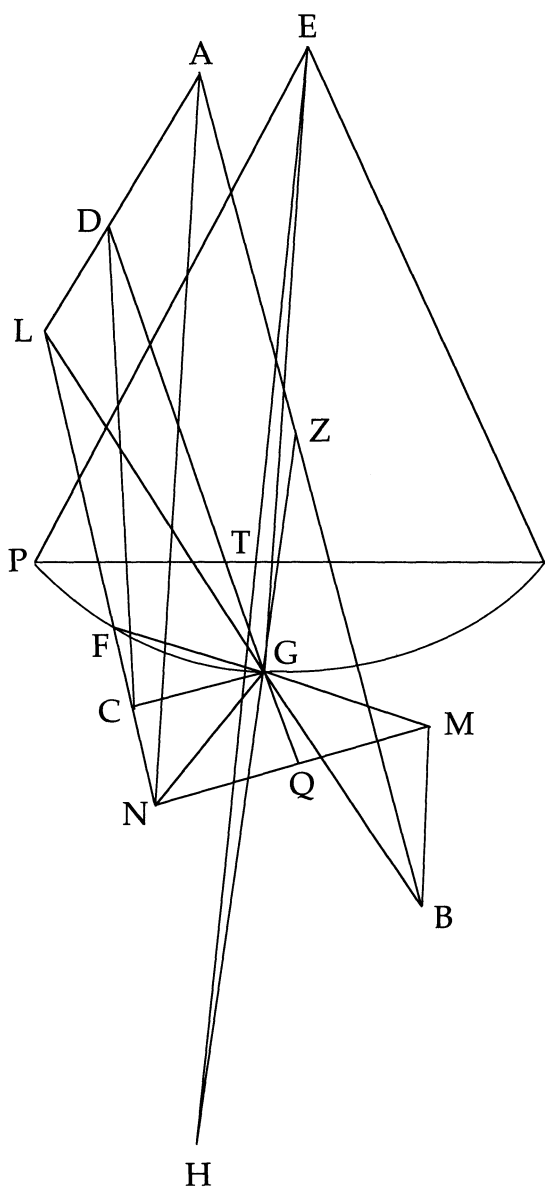


FIGURE 5.2.30

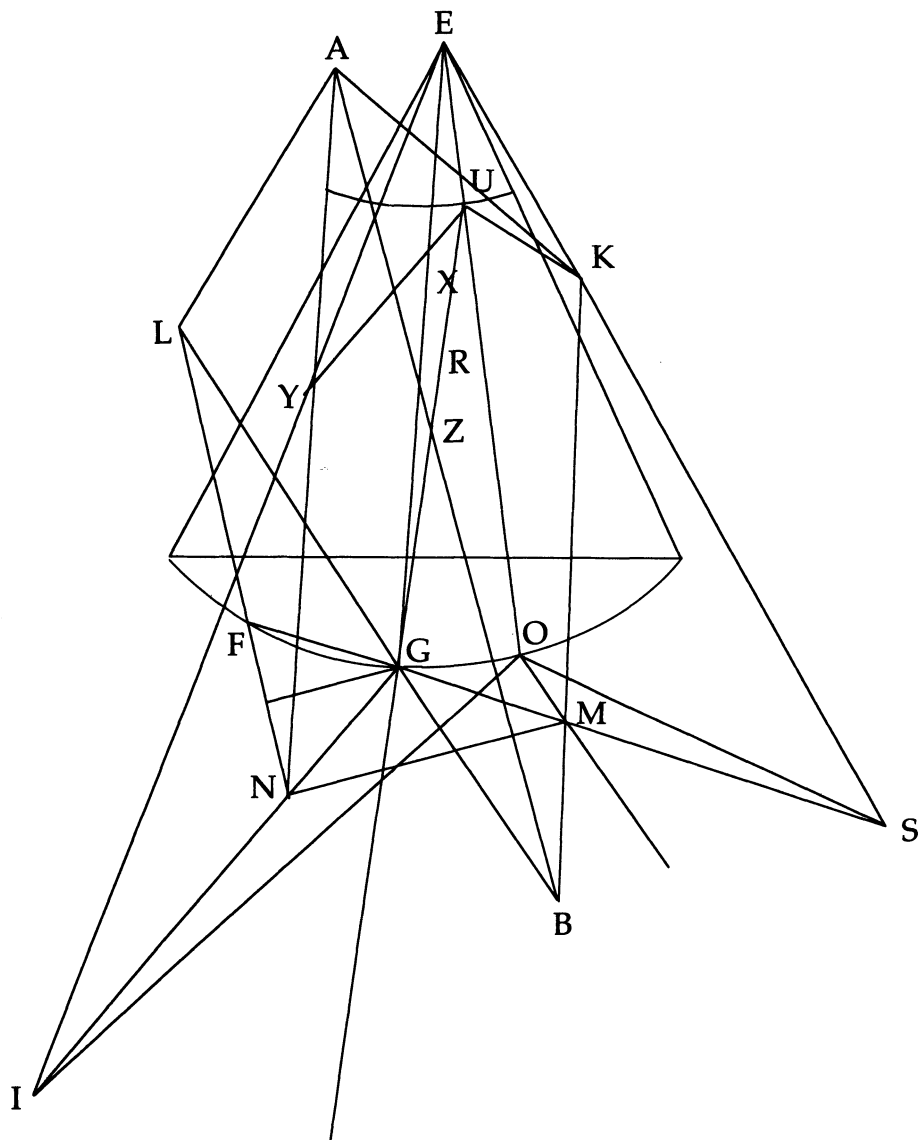


FIGURE 5.2.30a

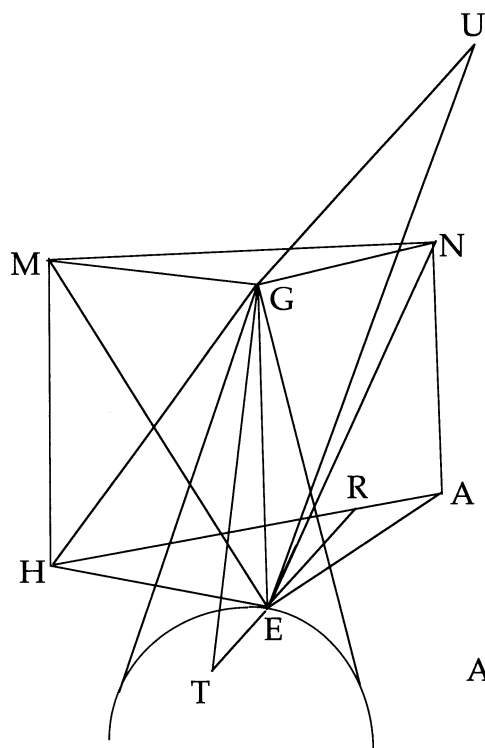


FIGURE 5.2.31a

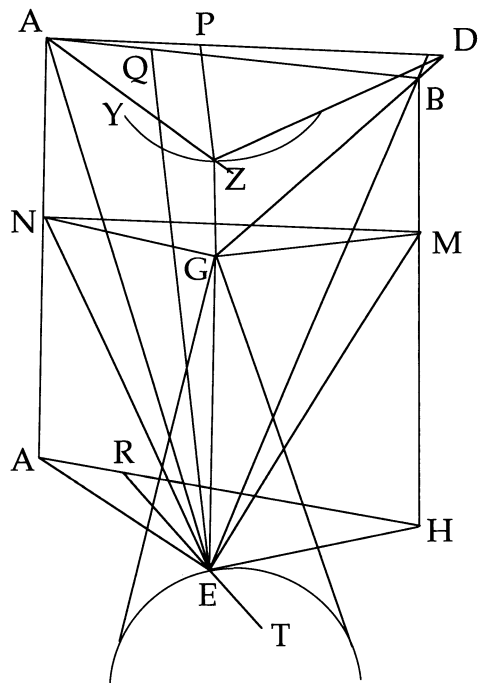


FIGURE 5.2.31b

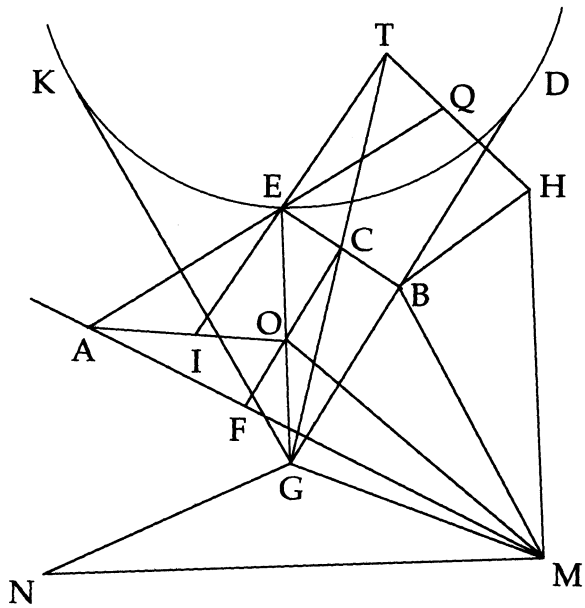


FIGURE 5.2.31c

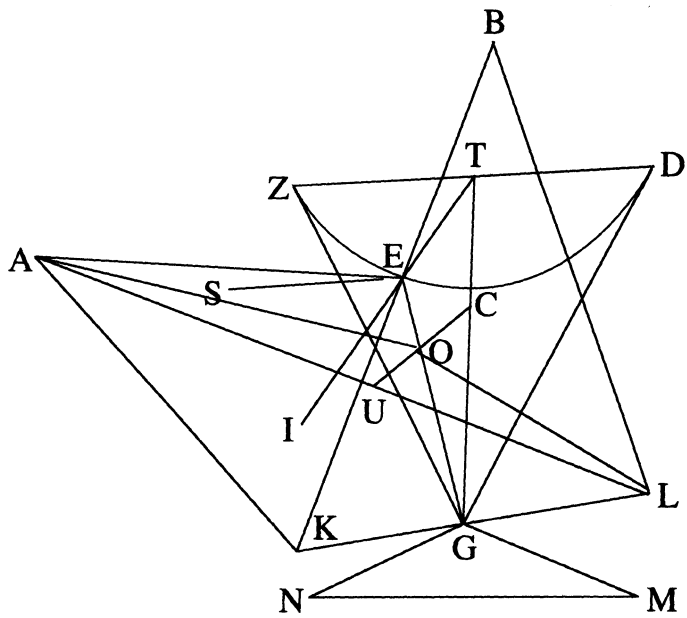


FIGURE 5.2.31e

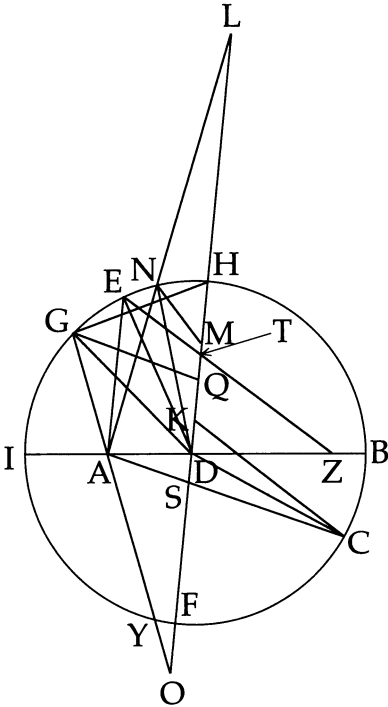


FIGURE 5.2.32

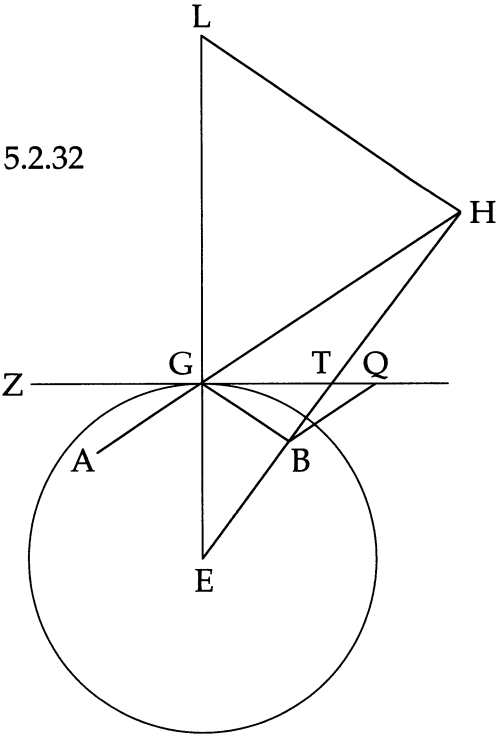


FIGURE 5.2.33

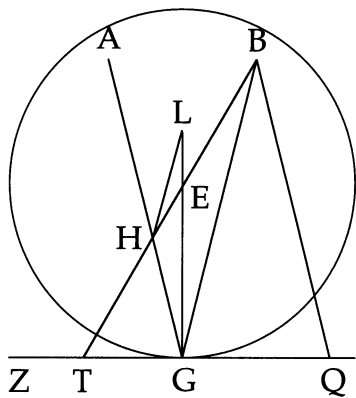


FIGURE 5.2.33a

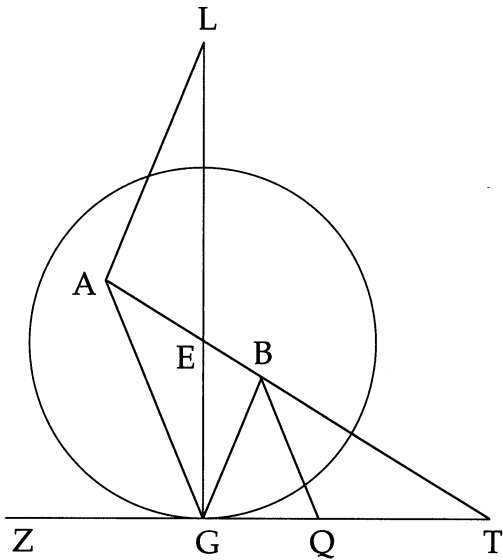


FIGURE 5.2.33b

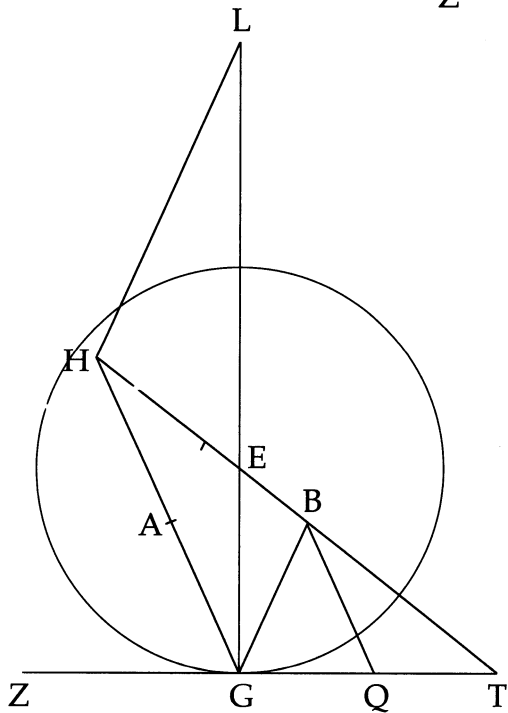


FIGURE 5.2.33c

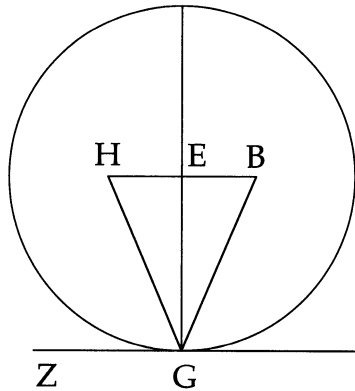


FIGURE 5.2.33d

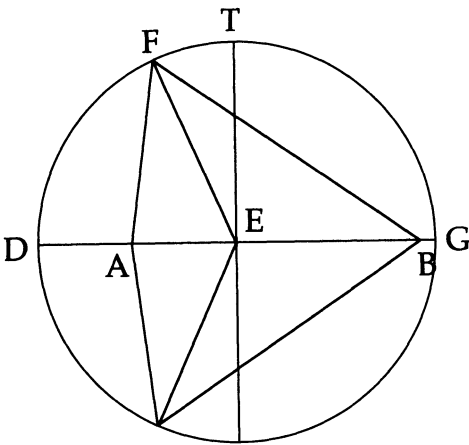


FIGURE 5.2.34

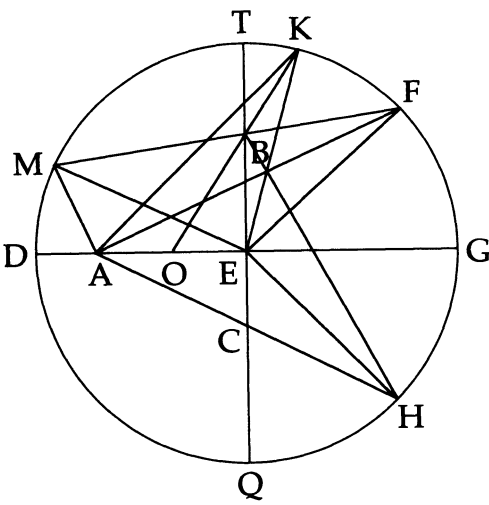


FIGURE 5.2.34a

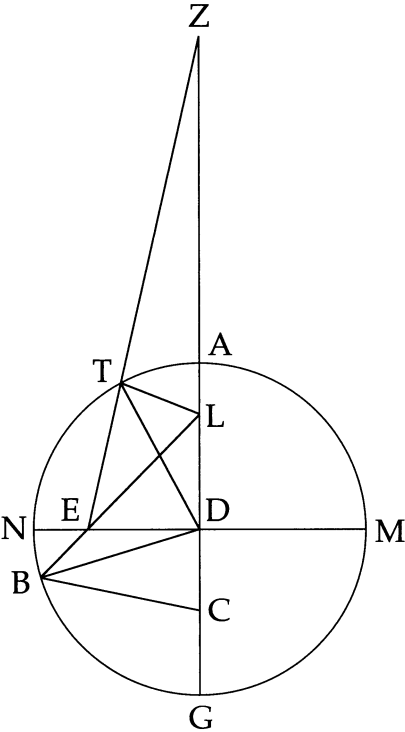


FIGURE 5.2.35

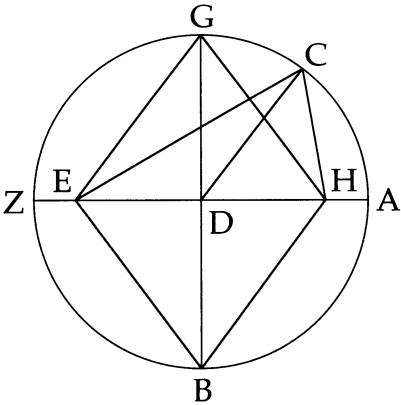


FIGURE 5.2.36

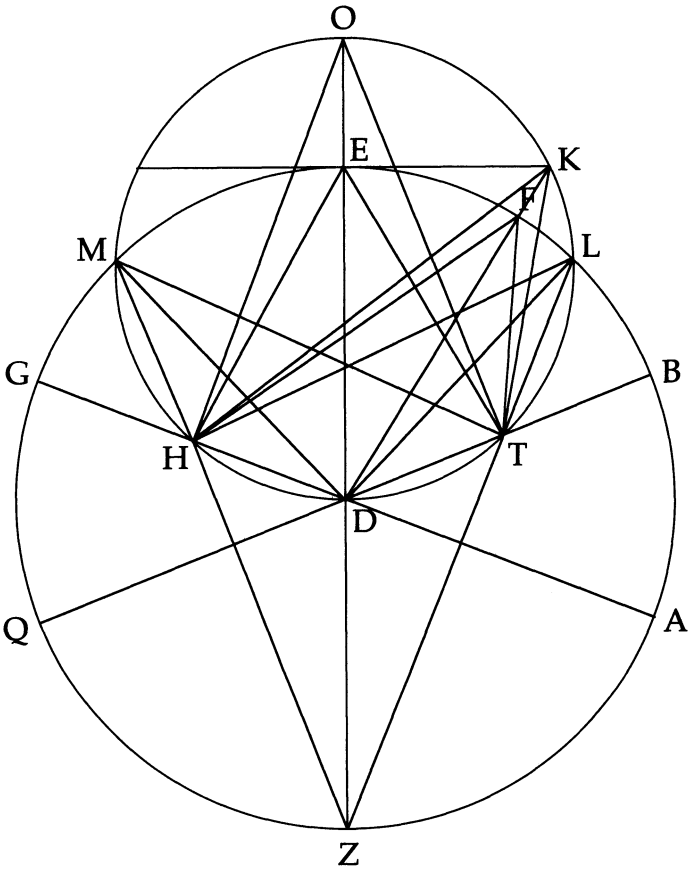


FIGURE 5.2.37c

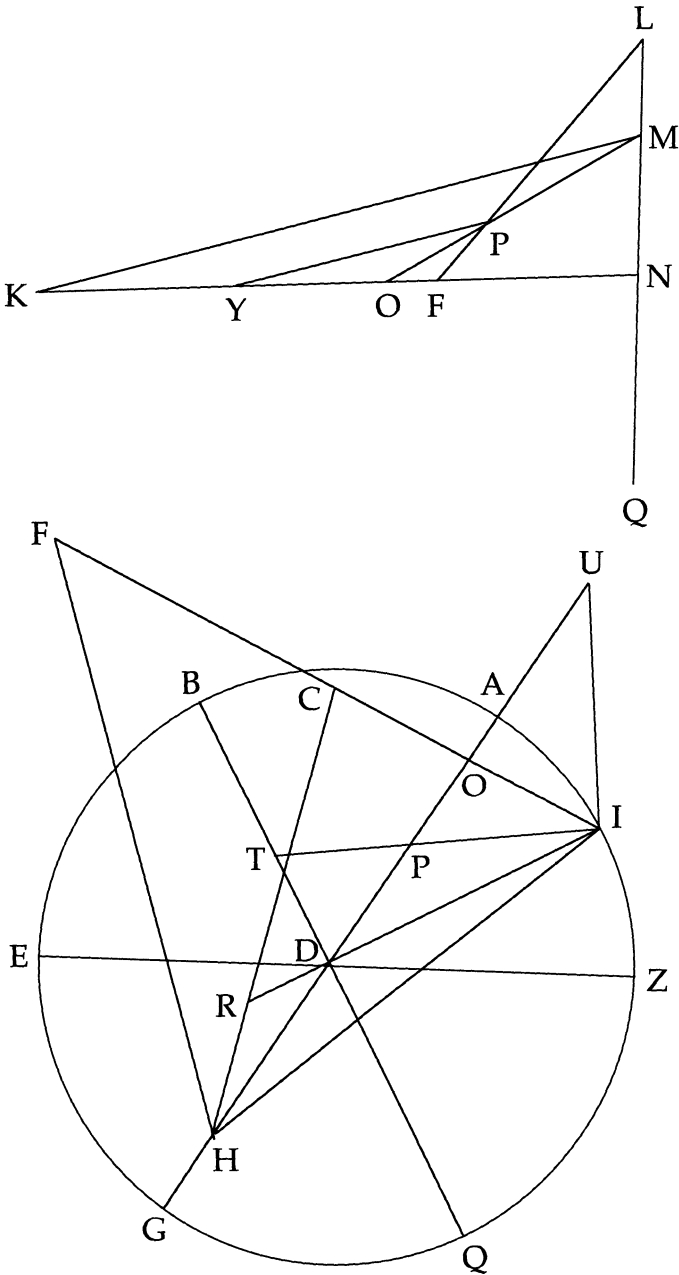


FIGURE 5.2.38

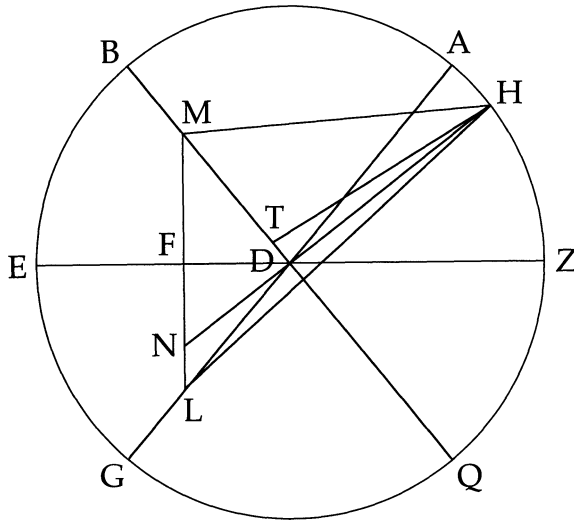


FIGURE 5.2.39

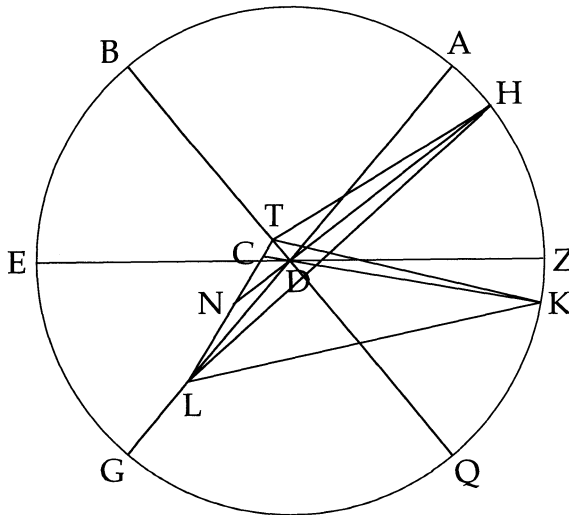


FIGURE 5.2.40

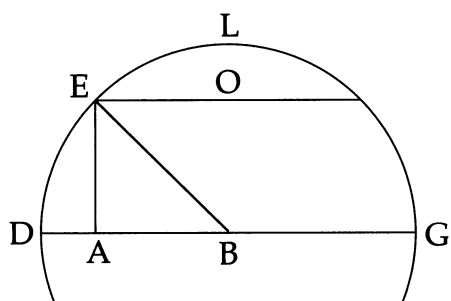


FIGURE 5.2.41

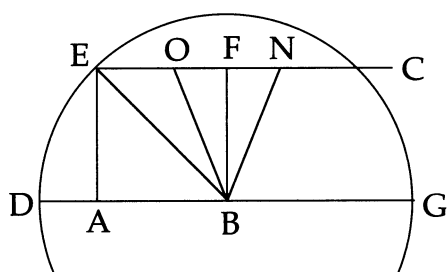


FIGURE 5.2.42

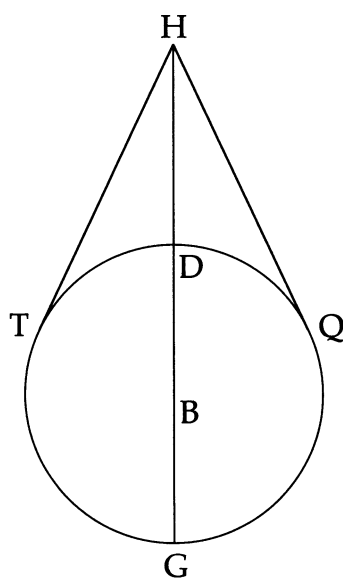


FIGURE 5.2.41a

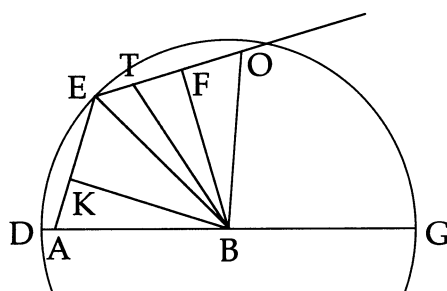


FIGURE 5.2.42a

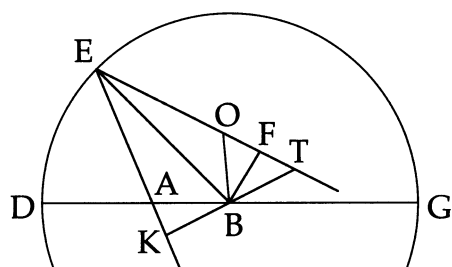


FIGURE 5.2.42b

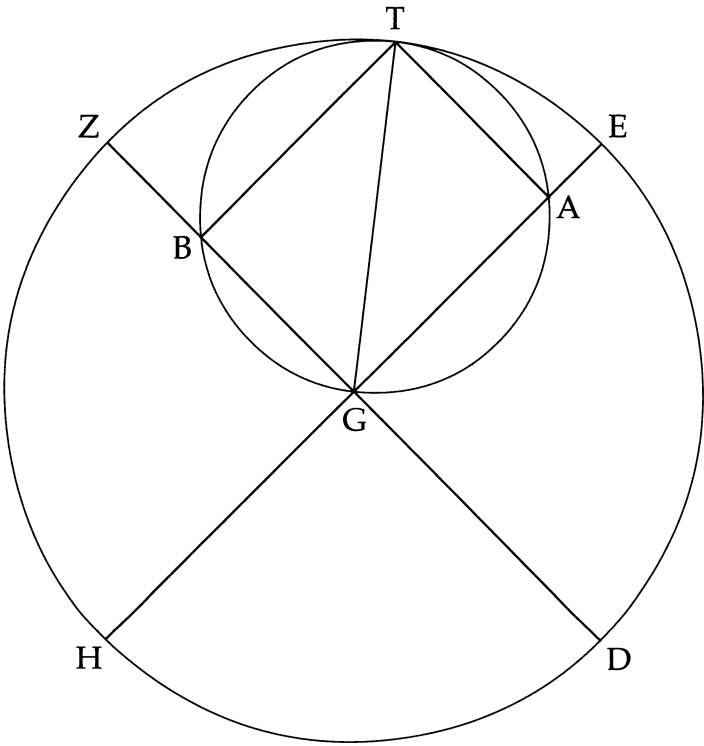


FIGURE 5.2.43

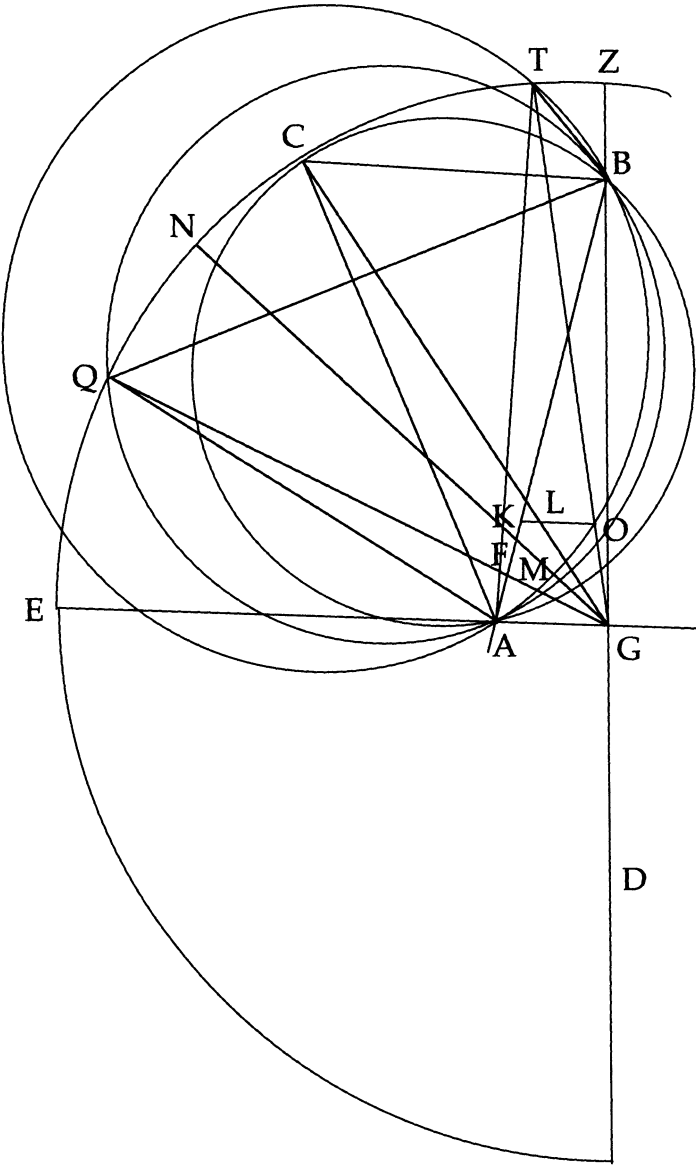


FIGURE 5.2.44

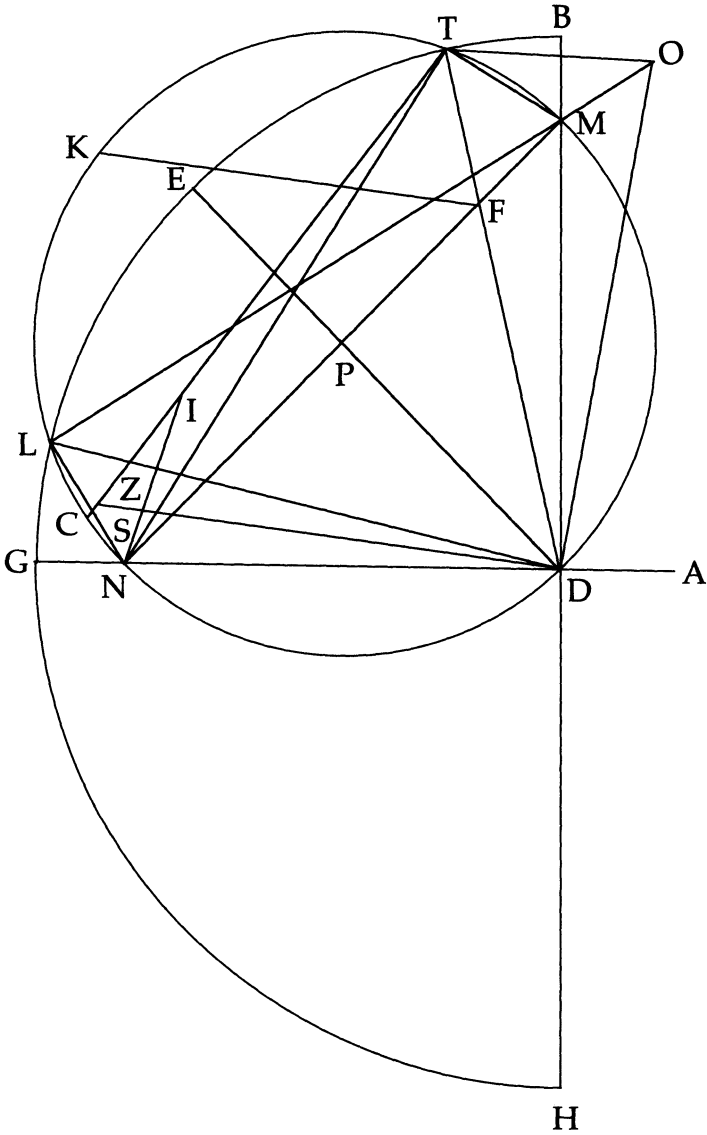


FIGURE 5.2.45

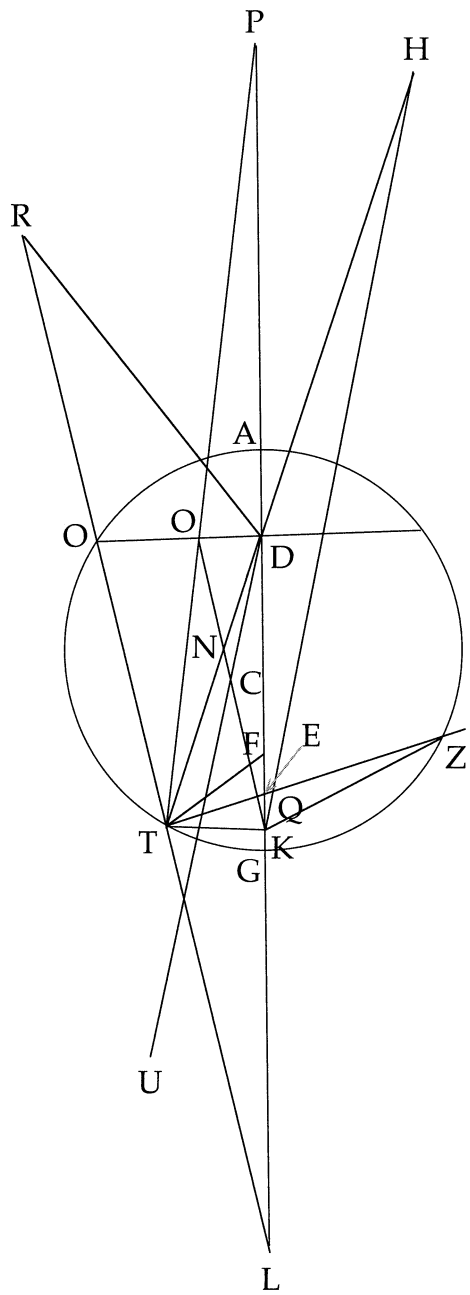


FIGURE 5.2.46

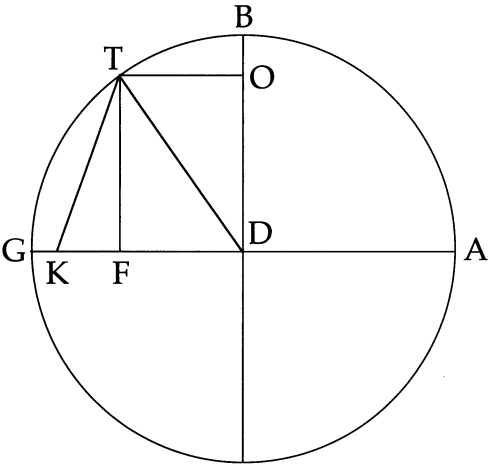


FIGURE 5.2.46a

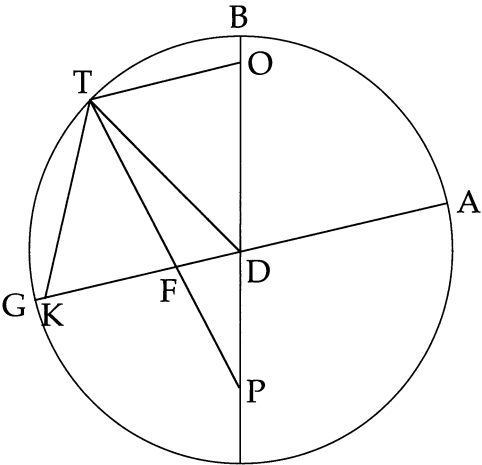


FIGURE 5.2.46b

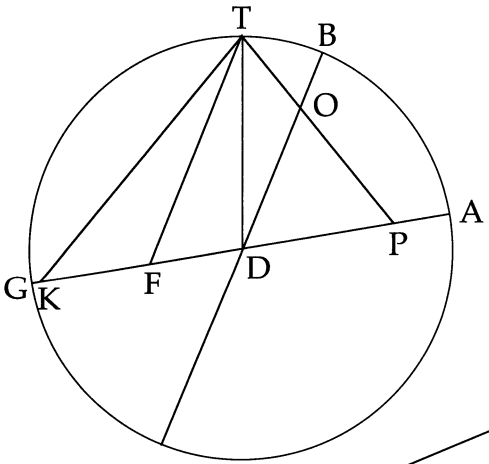


FIGURE 5.2.46c

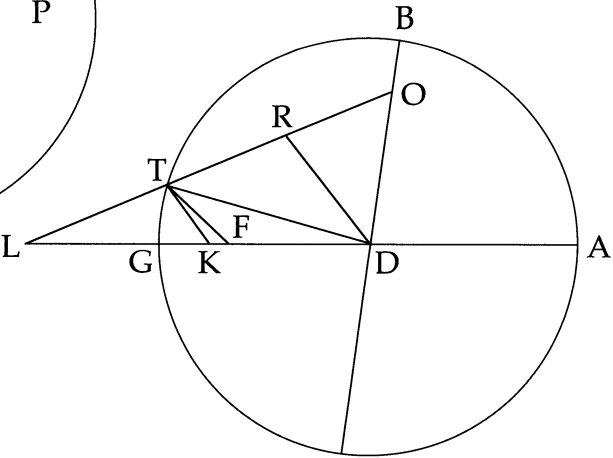


FIGURE 5.2.46d

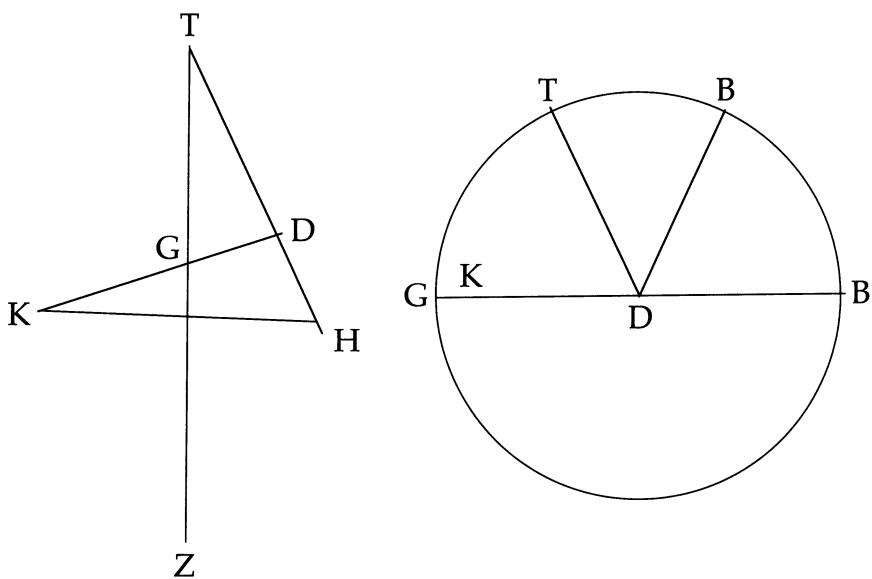


FIGURE 5.2.47

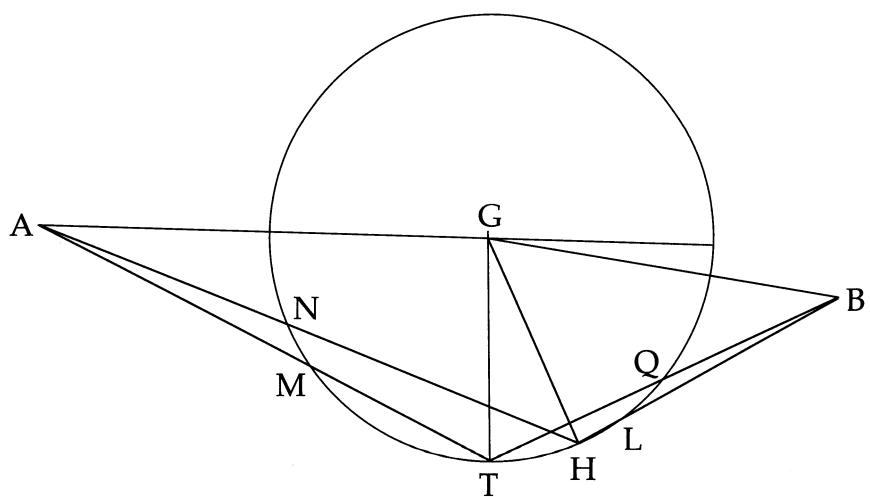


FIGURE 5.2.48

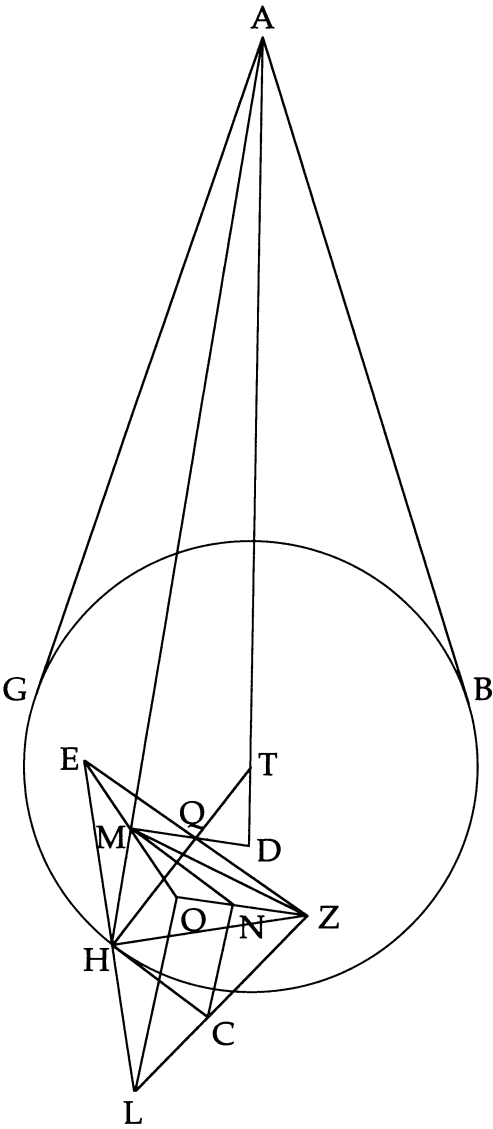


FIGURE 5.2.54

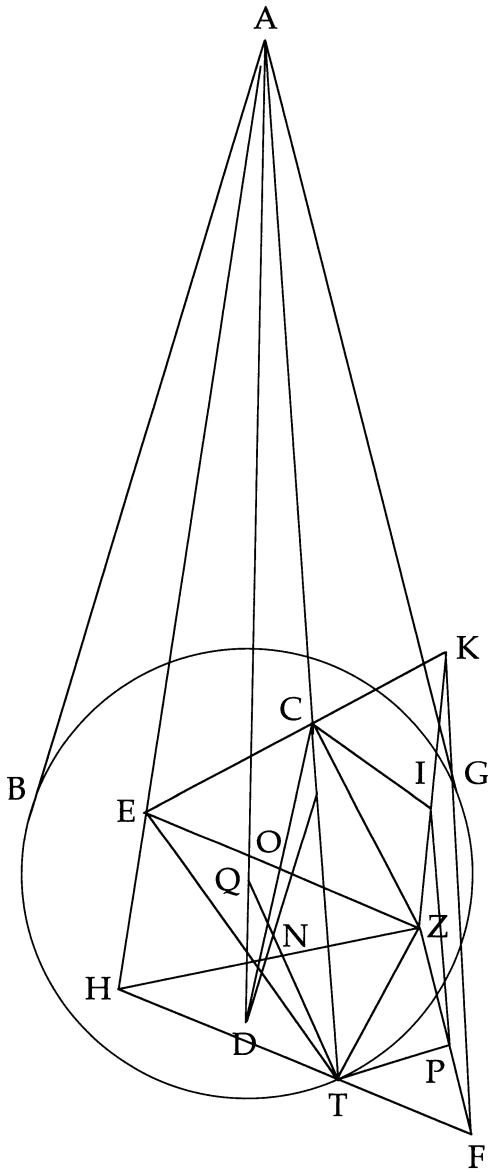


FIGURE 5.2.54a

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visus *frequently recurring* (3, 34, 37-55, 57-63, 65, 66, 68-70, 72, 74-91, 93-96, 99-103, 118, 121-124, 126-128, 131, 135, 136, 139-141, 143-148, 150, 151, 159, 162, 175-179, 182-184, 187) **center of sight, eye, sight, viewer, vision, visual faculty**

volere 9.165; 36.14, 21; 64.192; 76.298; 83.209; 175.4 **to mean** 300 **to want** 323, 393, 475 **to will** 346 **to wish** 323, 399

volvere 70.139 **to move (about)** 389

ydemptitas 24.291; 31.181; 41.143; 79.98; 80.130; 124.84 **equivalence** 313, 397 **identity** 327 **sameness** 319 **uniformity** 396, 432

ymago *frequently recurring* (63, 65-70, 72-83, 85-102, 122-127, 131, 143-145, 147-150, 152, 154, 155, 174-179, 182-184, 187) **image**

ypotesis 46.291; 91.115; 104.159; 109.289; 153.285; 155.36; 159.151; 162.233 **construction** 405, 416, 420, 455, 457, 463 **supposition** 331, 461

**ENGLISH-LATIN
GLOSSARY**

ENGLISH-LATIN GLOSSARY

a bit/a little bit modicus
above elatus
absence privatio
absolute inseparabilis
to absorb recipere
abutting contiguus
accidental/secondary accidentalis
to accomplish facere, peragere
account modus, ratio
account of reflection modus reflexionis
to account for determinare
accuracy veritas
actuality veritas
acute acutus
to add addere, adiungere
adjacent collateralis
to adjust mutare
air aer
alike similis
alternate colaternus
altitude altitudo
analysis modus
analysis of reflection modus reflexionis
angle angulus
angle of reflection angulus reflexionis
anterior anterior
apex caput
Apollonius of Perga Ablonius
apparatus instrumentum
apparent/appearing apparens
to appear apparere, videre
appearance apparentia
to apply adhibere, applicare, opponere
to apply before predicere

- to apprehend** *adquirere*
apprehension *adquisitio*
to approach *accedere, descendere*
appropriate *proprius*
arc *arcus, pars, sectio*
area *pars*
to argue before/previously *predicere*
to arise *accidere*
arrangement *dispositio, ordinatio*
to arrive *accedere, accidere, procedere, venire*
arrow *sagitta*
to ascertain *facere*
to assume *ponere*
to attach *adhibere, cogere, consolidare, figurare, infigere*
attachment *applicatio*
to attain *accedere*
to attribute *assignare*
axis *axis, dyameter*
- to balk** *repellere*
barley *ordeum (see also grain of barley)*
base *basis, caput*
base-block *tabula*
to be *accidere, manere*
to be apparent/evident/observed/seen/visible *apparere, patere*
to be as *habere*
to be bounded by *interiacere*
to be broader/larger/wider *excedere*
to be demonstrated/proven/shown *patere*
to be directly in line *opponere*
to be disparate *diversificare*
to be due to *accidere*
to be equidistant *equidistare*
to be established *patere*
to be far from *elongare*
to be formed *procedere*
to be hidden *latere*
to be incident *descendere, exire, venire*
to be invisible *latere, occultare*
to be left *remanere*
to be obligated *debere*
to be oblique *declinare*

- to be open** patere
to be parallel equidistare
to be rigid figere
to be tangent (to) contingere, tangere
to be unnoticeable occultare
beam radius
to bear in mind intelligere, notare
beautiful speciosus
beginning principium
below/lower inferior
between medius
beyond exitus, exterior
to bisect dividere, secare
bisection divisio
black niger, nigredo
block tabula
to block abscondere, obturare, occultare
blocking/facing disposition oppositio
body corpus, res
body of reflection corpus reflexionis
bolt repagulum
book liber
to bore extrahere
both duplex
bottom fundus, inferior
to bounce back revertere
bowed arcuatus
breadth latitudo
bright fortis
brightness maioritas
to bring together cogere
bronze eueus

to call nominare
to carry out facere
case situs
category partitio
cause causa
to cause efficere, facere, inducere
cavity concavatio, concavitas
celestis sky
center/centerpoint centrum, medius, punctum/punctus, punctus medius

- center of sight** centrum, centrum visus, visus
center of the eye centrum visus
change mutatio
to change auferre, mutare
chapter pars
character/characteristic proprietas, proprius, status
to choose sumere
circle circularis, circulus, spera
circular circularis
circumference circumferentia, latitudo
claim res
to claim affirmare
to claim earlier/previously predicere
clarity certitudo
clear lucidus, planus
to close claudere
to coincide applicare, cadere, continuare
coincidental continuus
color color, coloratio
to color colorare
coming together agregatio
common communis
common axis axis communis
common line linea communis
common section linea communis, sectio communis
compass circinus (*see also point of a compass*)
complement complementum
to complete complere
concave surface concavitas, concavum
to concentrate agregare
concentrated continuus
concentration agregatio
cone pyramidalis, piramis
conic/conical pyramidalis
conic section sectio pyramidalis (*see also hyperbola*)
to conjoin coniungere
to connect/join interiacere
consideration consideratio
to construct facere
construction ypotesis
contact contactus
to contain continere, includere

- continuation** continuus
to continue gradi, producere
continuity continuitas
continuous continuus
continuum continuitas, continuus
convex exterior
to convey vehere
to convey back mittere
to correspond respicere, respondere
corresponding similis
counterpart oppositus
course processus
to cover cooperire
to create facere
to cut secare
to cut off auferre
cutting sectio
cylinder columpna, columpnalis
cylindric/cylindrical columpnaris
cylindric section sectio columpnaris (*see also ellipse*)
- dark** fuscus, nigredo, tenebrosus
to darken obumbrare
deception error
to decrease diminuere
deduction sillogismus
to define facere, includere
definite proprius
degree mensura
to demonstrate accidere, demonstrare, ostendere, probare
demonstration demonstratio, probatio, ratio
depth altitudo, profunditas
to define assignare, determinare
to describe assignare, signare
to describe previously predicere
to detect videre
determination assignatio, certitudo, probatio
to determine assignare, certificare, debere, determinare, habere, invenire, perpendere, videre
to devote assignare
diameter dyameter, latitudo, quantitas
to differ diversificare

- difference** differentia, diversitas, proportio
different/not the same diversus, inequalis/inequaliter
digit digitus
dim modicus
direct directus, rectus
to direct adhibere, dirigere, figere, intueri, tendere
direction pars
directly facing rectus
directly in front oppositus
directly/right in line continuus
to disappear abscondere
to discuss exponere
to discuss previously predicere
discussion sermo
disparate diversus
to dispose accidere, adhibere, disponere
disposition dispositio, situs
disproof improbatio
to disprove improbare
distance altitudo, distantia, elongatio, longitudo, remotio
distinct diversus
to distinguish determinare
to divide dividere, partire
division divisio
to do facere, operari, peragere, preparare
to draw ducere, educere, erigere, extrahere, facere, producere, protrahere
to draw away elongare, remove
drill instrumentum
to drop descendere, ducere, erigere, producere

each/every thing singulus
edge acus, caput, extremitas, finis, latus, processus
to elevate elevare
elevated elatus
ellipse sectio columpnaris (*see also cylindric section*)
to emanate exire, procedere
to enclose includere
to encompass includere
to encounter invenire
end/endpoint caput, extremitas, finis, pars, summitas, terminus
to enter descendere, intrare, subintrare
entering descensus

- entry** gressus
enumerate numerare
to envelop occupare
to equal valere
equality equalitas, paritas
equivalence ydemptitas
to erase aufere
erect/perfectly erect ortogonalis
to erect erigere
erected erectus
error error
to establish (by demonstration) demonstrare, determinare, habere
to establish earlier predicere
estimation estimatio
even planus
evenness equalitas
evident planus
example verbum
excess excrementum
to exclude excludere
to exhaust finire
to explain assignare
explanation assignatio
to expose adhibere, exponere
to extend accedere, considerare, descendere, ducere, extendere, occurrere,
pervenire, procedere, producere, prolongere, protendere, protrahere, secare
to extend between interiacere
extension continuus
exterior extrinsecus
extrinsic accidentalis
eye oculus, visus
- face** facies
to face adhibere, declinare, opponere, respicere
facing oppositus
faint debilis, modicus
faintness debilitas
to fall between interiacere
to fall upon cadere, descendere, venire
familiar notus
far longus/longe, remotio
far enough modicus

to fashion facere

few rarus

fewness paucitas

figure figura

to fill extendere, obturare, occupare

find inventio

to find invenire

finger digitus

to finish complere

to fit adaptare, adhibere, applicare

to fit into intrare

fitting conveniens

to fix figere, infigere

fixed immotus

flat planitus, planus

to focus assignare

to follow accedere, accidere, cedere, mittere, movere, observare, operari,
patere, postponere

force motus, vis

to foremention predicere

form forma, modus

to form continere, efficere, facere, habere, includere, observare, tenere

form of a cone pyramidalitas

forming a rectangle ductus...in... (*see also* **rectangle**)

four-fold quadruplex

function proprietas

to gain acquirere

gap divisio

general universalis

generally universaliter

geometer geometer

geometrical geometricis/geometricus

to give sumere

to go procedere

goblet ciphus

grain of barley granum ordei

to grasp acquirere, comprehendere

great magnus

great circle circulus, circulus magnus

green viriditas

ground terra

- half** medietas, medius
hand manus
to happen accidere
to have/possess habere
hardness duricies
head caput
heavy honerosus, ponderosus
height altitudo, elevatio, eminentia, longitudo
held in place immotus
to hide occultare
to hold accidere
to hold up adhibere
hole foramen, vacuum
hollow (n.) concavitas, concavum, foramen
hollow (adj.) concavus
to hollow out fodere
horizon orizon
horizontality orizon
hyperbola sectio pyramidalis (*see also* **conic section**)
- identical** similis
identity ydemptitas
to illuminate dirigere, illuminare, mittere
illuminated lucidus
illusion error
image forma, ymago
image-location locus reflexionis, locus ymaginis
imaginary/imagined intellectualis, intelligibilis
to imagine intelligere, sumere
immobile immotus
imperceptable/imperceptably imperceptibilis/imperceptibiliter
to impinge descendere
to impress (upon) figere, imprimere, inficere, infigere
impression fixio, impressio
to impute assignare
incidence accessus, descensus, motus
to incline declinare
inclination declinatio
to include continere, includere
increase augmentum
indefinite/indefinitely infinitum, numerus

- individual** singularis
induction sillogismus
infinite infinitus
infinity infinitum
ink/red ink incaustum
inner/inside/interior inferior, interior
insensible insensibilis
to insert adhibere, descendere, immittere, intrare, ponere
instance verbum
instant hora
instrument instrumentum
to insure perpendere
intellectual intellectualis
intense fortis, magnus
to intensify redere, redire
intensity fortitudo
to interfere impedire
interference interpositio
interior segment interior
intermediate medietas
interrelationship intricatio
interruption discretio
to intersect concurrere, dividere, secare, tangere, transire
intersection concursus, sectio
intersection-point punctus sectionis
intrinsic proprius
invariable/invariably generalis/generaliter
iron ferreus
- to keep** manere
kind modus, proprietas, qualitas, quantitas, species
to know cognoscere, habere, scire
known manifestus, notus
- large** magnus
lathe instrumentum
latter posterior, postremus
to lay applicare, deprimere, descendere, statuere
to leave relinquere, remanere
length altitudo, distantia, longitudo, quantitas
lengthwise altitudo, longitudo
letter scriptura

- level** planities
to level inducere
to lie adhibere, cadere, existere, ponere, remanere
to lie between intercidere, interiacere, interponere
to lie far from elongare, remove
to lie outside egredi, preterere, preterire
light lux
likewise similiter
limit meta, terminus
limit-point terminus
line linea, protractio, via
line of longitude latus
line of reflection linea reflexionis
line of sight intuitus, linea visualis, radius visualis, visualis
to locate statuere
location locum/locus, pars, situs
location of reflection locus reflexionis, situs reflexionis
long longus/longe
longitude longitudo
to look (at/into) adhibere, inspicere, videre
to lower cadere, deprimere, descendere
luminous illuminosus, lucidus, lumenosus

magnitude magnitudo
to maintain conservare, durare, habere, observare, servare, tenere
maintained immutabilis
maintenance observatio
to make efficere, facere
to make beforehand/first precipere, preponere
to make out percipere
to make sure scire
man homo
manifest planus
manner modus
manual manus
mark mensura, nota, signum
to mark/mark off assignare, facere, notare, signare, sumere
mass moles
to mean volere
measure mensura
to measure/measure off metiri, signare, sumere
to meet concurrere

- method** modus
middle medius
midline linea media, medietas, medius
midpoint medietas, medius, punctus medius
to mingle admiscere, miscere
mirror speculum
to miss divertere
mode modus
mode of reflection modus reflexionis
moderate moderatus
moderate-size modicus
motion motus
motion in reflection motus reflexionis
to move descendere, movere
to move about volvere
to move away declinare
to move back retrocedere
to move forth procedere
movement processus
movement away recessus
movement toward accessus
to multiply multiplicare
- narrow** gracilis, strictus
to narrow constringere
natural naturalis
nature natura, naturalis
near propinquus
needle acus
neither neuter
to nest conserere
normal dyiameter, ortogonalis, perpendicularis
not to permit prohibere
noticable notabilis
- object** corpus, obiectio, res
object-point punctum/punctus, punctus visus
obliquity declinatio
to observe videre
obtuse obtusus
obvious manifestus, planus
to occupy occupare

- to occur** accidere, diversificare, facere
opacity soliditas
opaque solidus
open discoopertus
to open aperire
opening abscisio, foramen
operation res
to oppose opponere, respicere
opposed/opposite diversus, oppositus
orientation situs
origin/origin-point ortus
to originate incipere, oriri, procedere
orthogonal ortogonalis, perpendicularis, rectus
orthogonally ortogonaliter, perpendiculariter
other alteratio
outer (surface)/outside exterior
to outweigh addere, pondere
to overlap admiscere, concurrere

panel regula
parallel equidistans
parchment pargamentum
part/partial pars, partialis/partialiter
particular proprietas, singulus
to pass/to pass along/by/into/through descendere, incedere, intrare, movere, pervenire, procedere, secare, transire
to pass on redere, redire
to pass unseen preterere, preterire
passage transitus
passing through penetratio
peculiar proprius
peg baculus
to penetrate intrare, penetrare, transire
to perceive adquirere, comprehendere, percipere, sentire
perceptible perceptibilis
perception/process of perceiving adquisitio, comprehensio, perceptio
perfect finis
perpendicular ortogonalis, perpendicularis, rectus
person homo
to pertain accidere
phenomenon res
physical corporeus

place pars, situs

to place adhibere, applicare, facere, habere, ponere, statuere

to place between includere

to place upright erigere

plane planus, superficies

plank regula

plaque regula, tabula

to please libere

point acumen, locum/locus, pars, punctum/punctus, res, signum, situs, terminus

to point adhibere

point of reflection locus reflexionis, punctus reflexionis

point on a section punctus sectionis

point seen/viewed punctus visus

pointed acutus

pointer baculus

point of a compass pes circini

pole polus

polish politas, politio, politiva, politura

polished politus

polished body politiva, politum

polished surface politum

pore porus

portion pars, portio, sectio

position pars, situs

to position disponere

position below inferioritas

to precede predicere

to predict predicere

preliminary antecedens

to prescribe predicere

to present adhibere

to preserve observare

to press up to opponere

to prevent negare

principle principium

procedure modus, operatio, opus

to proceed movere, procedere

process iteratio

to produce ducere, efficere, facere, producere, referre

to project ducere, elevare, procedere, producere

to project outward egredi

proof probatio

to propagate ingredi, movere, occurrere, procedere

propagation motus

properness veritas

proportion proportio

to propose proponere

proposition propositio

to prove demonstrare, probare

proximity contiguitas

to punch perforare

quadrilateral quadrangulus

quantity quantitas

radial line radius

to radiate egredi, movere, procedere, venire

radiation accessus, processus

radius dyiameter, semidyiameter

to raise elevare, sublevare

rarely raro

ratio proportio

rationaly demonstrative

ray linea, radius

reach accessus

to reach accedere, accidere, cadere, continuare, descendere, movere, occurrere, pervenire, procedere, venire

reality veritas

to realize perpendere

reason causa

reasonable rationabilis

reasoning ratio

rebound reflexio, regressio

to rebound reflectere, regredi

rectangle ductus...in... (*see also forming a rectangle*)

rectilinear linea recta, rectilineus, rectitudo, rectus

red rubor

to reflect referre, reflectere

reflected reflexus

reflected angle angulus reflexionis

reflected ray linea reflexionis

reflection modus reflexionis, reflexio, reflexus

to relate to respicere

related group concursus
to remain manere, remanere
remainder residuus
to remove auferre, remove
to repeat iterare, replicare
to repel repellere
repetition replicatio
to replace collocare, ponere
repulsion repulsio
resistance fortitudo, prohibitio, repulsio
to restore reducere, regredi
result comprehensio
to return redere, redire, regredi
reversal revolutio
to reverse revolvere
revolution motus
to revolve movere
right directus, ortogonius, rectus
ring anulus
to rise ascendere
rod baculus
room domus
to rotate movere
rotation girus, motus
rough asperus
round circulus
to round circulari
roundness circulatio
ruler regula

same similis
sameness equalitas, ydemptitas
scientific understanding scientia
to scoop cavare
screen paries
to scrutinize adhibere
scrutiny intuitus
section linea, pars, portio, sectio
to see comprehendere, videre
to see that intelligere
seen apparens
segment pars, partialis, portio, sectio

semicircle semicirculus
to send back remittere
to sense sentire
sensible sensibilis, sensualis
sensible line/shaft linea sensualis
to separate remove, separare
to serve usitari
to set forth/out proponere
to set up adhibere, applicare, observare, statuere
setting-up erectio
scope via
shaft radius
shape figura, forma
sharp/sharp-edged or pointed acutus, rectus
sharp edge/point acuitas, acumen
to sharpen acuere
sharpness acuitas
to shift mutare
to shine accedere, cadere, declinare, descendere, mittere, procedere
to shine through intrare
shining descensus
show ostensio
to show ostendere, patefacere, probare
side latitudo, latus, pars
sight intuitus, oculus, visus
silver argenteus
similar similis
similarly similiter
situation situs
size magnitudo, mensura, quantitas
slant declinatio
to slant declinare
slight/slightly modicus
small exiguus, modicus, parvus
smallness parvitas
smooth lenis, politus
smoothness lenitas
softness mollities
solid solidus
source origo, ortus
spatial disposition situs
specific to proprietas

- to specify** determinare
sphere sphaera
spherical sphaericus
spot locum/locus, pars, punctum/punctus, situs
square quadratus
square (x^2) quadrangulus, quadratus
squared off orthogonalis
to squeeze out of shape immutare
to stand cadere, erigere, statuere
standing erectus
stationary immotus
stone lapis
stop obstaculus
straight directus, rectus
straight line linea recta
to strike cadere
strong fortis
to subdivide dividere, subiacere
to subtend intercidere, interciperere, respicere, tenere
to subtract auferre, removere
to sum up to valere
sun sol
sunlight sol
to suppose ponere
supposition ypotesis
surface altitudo, facies, pars, superficies

tablet regula, tabula
to take habere, sumere
to take out extrahere
to take place facere
template tabula
to tend (in a direction) tendere
theorematically demonstrative
theory opinio
thick spissus
thickness latitudo, spissitudo
thin gracilis, subtilis
thing res
to tilt downward declinare
time hora
tiny modicus

- token** modus
top caput
to touch cadere, contingere, descendere, secare, tangere
to transect secare
to transfer transfere
transparency raritas
transparent rarus
trap fixio
to travel procedere
triangle triangulus
triangular triangularis
to truncate secare
truth fides, veritas
tube columpna
to turn inclinare
to turn out evenire
twofold duplex
- uncovered** discoopertus
to understand intelligere, scire
unequal inequalis/inequaliter
uniformity similitudo, ydemptitas
uniformly equidistans
unity unitas
untouched vacuum
upright erectus, ortogonalis, rectitudo, rectus
- variation** diversitas
variety varietas
various diversus
to vary diversificare
veridity veritas
to verify verificare, videre
vertex acumen, caput, conus
vessel concavum, vas
vice-versa e conversus
to view adhibere, comprehendere, inspicere, videre
viewed object/visible body corpus visum
viewer visus
viewpoint intuitus
visibility apparentia
visible apparens

visible object res, res visa

visible point punctus visus

vision visus

visual visualis

visual axis dyiameter visualis, linea visualis, perpendicularis visualis

visual faculty visus

visual plane visualis superficies

void vacuum

wall paries

to want volere

water aqua

wax cera, cereus

way modus

weak debilis

to weaken debilitare, diminuere, subtrahere

weakening debilitas, debilitatio, minoritas

white albus

width latitudo, quantitas

to will volere

window foramen

to wish volere

to wobble vacillare

wooden ligneus

wooden rod lignum

writing scriptura

to yield pretendere

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This is a continuation of Mark Smith's publication of Ibn al-Haytham's *De aspectibus*, the most important treatise on optics of the Middle Ages, indeed, the most important treatise prior to the seventeenth century. A. I. Sabra has edited and translated Books 1-3 of the Arabic text. The Latin manuscript text, published by Smith, the text known in Europe prior to Risner's edition of 1572, differs notably from the Arabic and from Risner's edition, and requires independent publication; an annotated translation is necessary to make the work useful to as many readers as possible. Smith's edition appears definitive and his translation a model of accuracy and clarity; the introduction and notes contain all that is necessary to elucidate this difficult, technical work. In every way, this is an exemplary publication.

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The edition of Alhacen's text will appeal to those who are interested in medieval scientific developments, but also to others for whom the study of optics, in particular, forms part of the cultural development of the later Middle Ages. The understanding of vision continues to gain importance in the historiography of intellectual life in this period, and Mark Smith's work will be an important contribution to this scholarship. Professor Smith is well versed in the literature of his subject and has not missed an important discussion of Alhacen and optics in the secondary literature. His translation of the Latin text accurately represents a challenging and technical original.

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